How Equilibrium Prices Reveal Information in a Time Series Model with Disparately Informed, Competitive Traders*

Todd B. Walker†

January 2007

Abstract. Accommodating asymmetric information in a dynamic asset pricing model is technically challenging due to the problems associated with higher-order expectations. That is, rational investors are forced into a situation where they must forecast the forecasts of other agents. In a dynamic setting, this problem telescopes into the infinite future and the dimension of the relevant state space approaches infinity. By using the frequency domain approach of Whiteman [41] and Kasa [19], this paper demonstrates how information structures previously believed to preserve asymmetric information in equilibrium, converge to a symmetric information, rational expectations equilibrium. The revealing aspect of the price process lies in the invertibility of the observed state space, which makes it possible for agents to infer the economically fundamental shocks and thus eliminating the need to forecast the forecasts of others.

Keywords: Asset Pricing, Asymmetric Information, Higher-Order Beliefs

JEL Classification Numbers: G12, D82

*I would like to thank Matthew Billett, John Geweke, M. Beth Ingram, Ken Kasa, Narayana Kocherlakota, Matthew Pritsker, B. Ravikumar, Gene Savin, Gábor Virág, and participants at New York University, the Federal Reserve Board of Governors, Indiana University, Miami University, the University of Iowa Alumni Conference, the Midwest Macroeconomics Conference, and the Midwest Economic Theory Group for their helpful comments and suggestions. I would like to thank an anonymous referee for many incisive comments. Finally, I would to thank my advisor, Chuck Whiteman, for several insightful discussions. All remaining errors are my own.

†Department of Economics, Indiana University, 100 S. Woodlawn, Bloomington, IN 47405, *E-mail*: walkertb@indiana.edu *Phone*: 812-855-1021 *Fax*: 812-855-3736

1 Introduction

Keynes [21] likened investment decisions to beauty contests in which competitors had to select the "prettiest faces" from a hundred photographs. Keynes argued that competitors will not pick the faces they find most attractive but will be forced into a situation where they must guess what average opinion expects the average opinion to be; that is, they must form higher-order expectations. This scenario accurately describes the behavior of differentially informed investors in a simple asset pricing model. Traders recognize that information disseminated through current and past prices reflects not only the information endowment of other investors, but also their attempts to forecast the forecasts of other traders. The goal of this paper is to explicitly model how information is disseminated through prices, and to show when this dissemination becomes too widespread to maintain asymmetric information in equilibrium. Models previously believed to impose asymmetric information in equilibrium (e.g., [35]) are shown to converge to symmetric information, representative agent models.

The empirical failures of representative agent models have inspired a vast literature that introduces information heterogeneity and higher-order expectations in hopes of reconciliating theory with data.[1] However, the computational problems associated with heterogeneous information have long proven to be a significant impediment to solving dynamic, intertemporal trading models with differentially informed investors. The crux of the problem is how to model higher-order expectations. Investors who anticipate what average opinion expects the average opinion to be form, what Keynes called, third-degree expectations. Due to the complexities of the fourth degree, Phelps [30] quipped "one gets a vertiginous feeling, the eyes dull, and the face goes slack." Yet in a dynamic model, rational investors will fall into the trap of *infinite* regress where they must forecast what the average opinion expects the average opinion expects the average opinion..., *ad infinitum*. Agents behaving optimally must incorporate these infinite-order expectations into their beliefs, thus making the state space infinite dimensional.

To bypass the technical difficulties associated with higher-order belief dynamics, the literature has invoked clever assumptions.[2] For example, as described by Kasa [19], Lucas [23] was the first to formally encounter the problem of infinite regress and simply assumed a pooling equilibrium in which firms exchanged information each period, thus eliminating higher-order beliefs in equilibrium. Townsend [36] resorted to a truncation strategy, where the state of the economy is revealed to all agents with a two-period lag. By truncating the infinitely dimensional state space to finite dimensions, Townsend [36] could make use of standard solution techniques. Singleton [35] was the first to study a dynamic asset pricing model with higher-order belief dynamics and also employed the truncation technique of Townsend to solve the model.[3] In contrast, this paper does not assume away the infinite-regress problem. Working within the context of Singleton's bond market model, this paper gives an

[1] Recent contributions include [1, 4, 3, 6, 8, 18, 27, 37, 43].

[2] Approaches to elude the problem of infinite regress have included: constructing a hierarchical information structure with limited dispersion of information (e.g., [40], [39]); assuming a continuum of investors and invoking the law of large numbers to remove all aggregate uncertainty (e.g., [17]); assuming a finite or static environment in which the asset is liquidated at a certain date (e.g., [3], [14]).

[3] More recently, Bacchetta and van Wincoop [5] use Townsend's truncation technique to analyze exchange rate dynamics.

explicit solution procedure that does not rely on truncating the state space. Frequency domain techniques are shown to provide sharp analytical characterizations of equilibrium processes, and invertibility of the state space is shown to determine if higher-order beliefs can be preserved in equilibrium.

While couched within the context of an asset pricing model, the techniques introduced here contribute more broadly to the literature on solving dynamic models with asymmetric information. This paper builds on the work of Sargent [34], Kasa [19], and Pearlman and Sargent [29]. Sargent describes a method for computing equilibrium (vis-á-vis Townsend's partial revelation approach) in which the state of the economy is never revealed to the agents. Kasa demonstrates how the frequency domain significantly decreases the computational cost of Sargent's solution method. More recently, Pearlman and Sargent show that the signal extraction problem of Townsend [36] is not enough to preserve divergent beliefs when agents act rationally. By defining a new state variable (the agent's forecast error) and applying the recursive methods of Pearlman et al. [28], they show that every agent will make the same forecast in equilibrium, thus eliminating the need to forecast the forecasts of others.

These contributions ([34], [19], [29]) have all been made from examining the model of Townsend [36]. This is an important distinction because higher-order belief dynamics in Townsend's model rely entirely on an assumption of lags in production, which prohibit prices from simultaneously clearing markets. Kasa [19] refers to the resulting signal extraction problem faced by firms as "endemic" to the economy of Townsend and unlike the "typical" signal extraction problem–i.e., where there are more noises than signals. In contrast to Townsend, the bond market model of Singleton [35] introduces higher-order belief dynamics by assuming investors must solve a "typical" signal extraction problem. Singleton Singleton follows Townsend in approximately solving the model by truncating the state space, but in his concluding remarks, Singleton suggests that higher-order beliefs seem to play a limited role in equilibrium:

> "Another interesting finding is that the equilibrium prices for the models with disparate information and partial, homogeneous information follow very similar time series properties. It remains to be seen whether this carries over to alternative parameterizations and information structures. Based on the findings to date, however, it appears that disparate information per se in a competitive market does not significantly effect the equilibrium price process."

This paper demonstrates how higher-order beliefs play no role in equilibrium and contributes to the literature in several ways. First, by examining the "typical" signal extraction problem faced by investors in Singleton's bond market model, this paper fills a gap in the literature and extends the findings of Kasa [19] and Pearlman and Sargent [29] to nonspecialized environments. Second, in contrast to the methodology of Pearlman and Sargent, I show how Hilbert space methods and frequency domain techniques can be employed to prove that endogenous variables reveal all privately held information. The result assumes agents have basic econometric knowledge and a simple invertibility condition determines if higher-order beliefs can be sustained in equilibrium. Third, I examine the *non-truncated* version of Singleton's model, which speaks to the suggestion of Pearlman and Sargent to reinstate the forecasting the forecasts of others problem by "increas[ing] the dimension of decision makers' private information relative to the number of endogenous variables (prices) that decision makers can condition their forecasts upon." The results established here add precision to this statement in that a significant decrease in the signal-to-noise ratio is shown not to be enough to preserve higher-order beliefs. Fourth, I am able assess the approximation error associated with truncating the state space by comparing the equilibrium found without truncation to that of Singleton [35]. In doing so, this paper proves the conjecture that higher-order beliefs play no role in equilibrium. Finally, given the technical challenges mentioned above, a majority of the work in this area relies on numerical solution methods. This paper lays out an explicit solution procedure for solving dynamic models with asymmetric information *analytically*. The results derived here have implications for static models of asymmetric information, models that assume agents have limited information capacity, and models that use the Townsend-Singleton truncation technique.

The next section introduces the bond market model of Singleton. Section 3 provides a solution method for dynamic models of asymmetric information, solves the model in a symmetric information setting and in an asymmetric information environment, and proves that these two information structures yield the same equilibrium.

2 The Model

In this section, I briefly outline the model of Singleton [35], which was motivated by the market microstructure of the U.S. bond market.[4] The basic features of the model include a competitive, Walrasian market structure with a single asset that is traded among speculative and nonspeculative or liquidity traders.

Suppose there is a continuum of investors indexed by $i \in [0,1]$, and each trader invests in a single risky security with price p_t and stochastic coupon payment c_t at date t. The coupon stream $\{c_t\}$ is assumed to be normally distributed and to follow a first-order autoregressive process

$$c_t = \bar{c} + \psi c_{t-1} + u_t, \qquad \mathbb{E}(u_t) = 0,\ \mathrm{var}(u_t) = \sigma_u^2,\ \ |\psi| < 1.$$

Purchases of the security are financed by borrowing at the constant rate r. Therefore, the wealth of trader i evolves according to

$$w_{i,t+1} = z_{it}(p_{t+1} + c_{t+1}) - (1+r)(z_{it}p_t - w_{it}),$$

where z_{it} denotes the holdings of the risky asset at date t. The ith investor is assumed to have a one-period investment horizon and to rank alternative investment strategies according to the negative-exponential utility function

$$\mathbb{E}_t^i - \exp(-\gamma w_{i,t+1}),$$

[4]In addition to the reasons listed in the Introduction, the model of Singleton is of recent interest due to the work of Allen, Morris and Shin (hereafter AMS; see, [25], [3]). The AMS framework is a static, finite-horizon asset pricing model. The techniques of Singleton can be used to extend the AMS setup to an infinite horizon economy.

where $\mathbb{E}_t^i$ denotes the expectation of investor i conditioned on his information set Ω_t^i at date t, and γ is the coefficient of absolute risk aversion that is common to all traders. Given the assumption that underlying sources of uncertainty are normally distributed, the conditional expectation can be calculated from the (conditional) moment generating function for the normally distributed random variable $-\gamma w_{i,t+1}$. That is,

$$-\mathbb{E}_t^i \exp(-\gamma w_{i,t+1}) = -\exp\{-\gamma \mathbb{E}_t^i(w_{i,t+1}) + (1/2)\gamma^2 \mathrm{v}_t^i(w_{i,t+1})\}$$

where v_t^i is the conditional variance and $\mathrm{v}_t^i(w_{t+1}) = z_{it}^2 \mathrm{v}_t^i(p_{t+1} + c_{t+1})$. Stationarity implies the conditional variance term will be a constant; that is define $\mathrm{v}_t^i(w_{i,t+1}) \equiv z_{it}^2 \delta_i$. The agent's demand function for the risky asset follows from the first-order necessary condition for maximizing expected wealth, i.e.,

$$0 = -\gamma\left[\mathbb{E}_t^i(p_{t+1} + c_{t+1}) - (1+r)p_t\right] + \gamma^2 z_{it}\delta_i$$

and thus the agent's demand for the risky asset is given by

$$z_{it} = \frac{\mathbb{E}_t^i(p_{t+1} + c_{t+1}) - \alpha p_t}{\gamma \delta_i} \tag{1}$$

where $\alpha = (1+r) > 1$. It is the presence of higher-order moments δ_i that lead to multiple equilibria (see, [24] and [38]). Appendix A shows that while the main results of the paper continue to hold, strict parameter restrictions must be imposed to solve the nonlinear model analytically. This problem is commonplace in the literature and in order to avoid uniqueness and multiplicity issues, the conditional variance term is typically assumed to be constant and common to all traders (e.g., [11]). In the model studied below, I normalize the coefficient of risk aversion and the variance parameter to unity. The resulting demand schedule

$$z_{it} = \mathbb{E}_t^i(p_{t+1} + c_{t+1}) - \alpha p_t \tag{2}$$

not only offers a unique equilibrium, but also has a broad appeal in that there are many economic models that have equations analogous to that of (2). For example, with c_{t+1} defined as dividends, we have a present value model for stock prices; with c_{t+1} defined as the difference between national money supplies and income levels, it becomes the monetary model of exchange rates; with c_{t+1} defined as a short-term interest rate, it becomes the expectations hypothesis of the term structure; and so on.

Singleton assumed supply of the asset was stochastic and determined by nonspeculative traders, which serves to break the no-trade theorem. That is, the *net* supply of the asset s_t (total supply less nonspeculative demand at time t, less the mean difference) is the sum of two stochastic components and price

$$s_t = A(L)\varepsilon_{1t} + B(L)\varepsilon_{2t} + \xi p_t \tag{3}$$

where $A(L)$ and $B(L)$ are (possibly infinite-order) polynomials in nonnegative powers of the lag operator L with square-summable coefficients (i.e., $\sum_{j=0}^{\infty} A_j^2 < \infty$), and $\{\varepsilon_{1t}\}$ and $\{\varepsilon_{2t}\}$ are mutually and serially uncorrelated, normally distributed random variables with zero mean and variance components $\sigma^2_{\varepsilon_1}$, $\sigma^2_{\varepsilon_2}$.[5] The net supply is interpreted as arising from non-speculative traders in the U.S. bond market (e.g., the U.S. Treasury, the Federal Reserve, financial intermediaries). The shocks to net supply could thus arise from non-speculative traders attempting to satisfy "macroeconomic" objectives and for technical reasons related to the financial intermediation process. Non-speculative traders are assumed to respond positively to an increase in price; thus, following Singleton, I will assume $\xi > 1$.

The market clearing condition equates equations (2) and (3), and gives the equilibrium price to be

$$\varphi p_t = \int_0^1 \mathbb{E}_t^i p_{t+1} di + \bar{c} + \psi c_t - s_t \tag{4}$$

where $\varphi = (\xi + \alpha) > 1$. Thus, the equilibrium price of the asset at time t depends upon the market-wide average expectation of the asset price at $t+1$.[6] However, each trader's forecast of p_{t+1} will depend upon the market-wide forecast of p_{t+2}, and so on, *ad infinitum.* If traders' information sets generate disparate expectations, the problem of infinite regress arises. When forecasting p_{t+1}, each trader must take into account every other traders' forecast of $p_{t+1}, p_{t+2}, \cdots$. If the equilibrium price does not reveal all privately held information, then the usual method of solving for the rational expectations equilibrium (e.g., [9]) breaks down.[7] The principal technical difficulty is that agents are extracting signals from *endogenous* variables. When endogenous variables convey information, it becomes difficult to identify a tractable set of state variables because an agent's notion of the state of the economy would include other agents' forecasts of the asset price at indefinitely many future dates, thereby making the dimension of the state indefinitely large. In contrast, for symmetric information, representative agent models involving signal extraction from endogenous variables, the true state vector is latent from the optimizer. *Beliefs* then serve as the defacto state variable (e.g., [26]), and the Kalman filter can be employed to estimate the hidden state. But when the state becomes infinite dimensional, as it does here, this method cannot be applied.

The next section describes a solution procedure for handling the infinite-dimensional state space problem faced by the investors in Singleton [35]. In lieu of truncating the state space, the solution technique uses frequency domain methods to analytically solve for equilibrium. The solution procedure also gives conditions under which the equilibrium price process contains too much information to preserve higher-order beliefs in equilibrium.

[5]Eq. (3) places no restrictions on the serial correlation properties of $\{s_t\}$. The Wold Decomposition Theorem allows for such a general structure (see, [33]).

[6]Notice that (4) does not imply asymmetric information with respect to the coupon stream. The argument is that asymmetric information in the U.S. bond market would arise with respect to the supply process.

[7]Moreover, as recently emphasized by [3], average expectation operators usually fail to satisfy the law of iterated expectations. Therefore, in an economy with asymmetric information, the price of an asset today will *not* equal the representative agent's discounted expected value of the asset's payoff stream conditional on information available today, but the price will also encompass investors' higher-order beliefs (see also, [7].

3 Information and Solution Techniques

3.1 Information

It is important to be meticulous about the conditioning information sets of agents in models of asymmetric information, especially when information can be extracted from endogenous variables. To place structure on the problem, solutions to the model will be sought in the space spanned by square-summable linear combinations of the fundamental driving processes $\{u_t, \varepsilon_{1t}, \varepsilon_{2t}\}$. This assumption rules out sunspot equilibria and implies that the solution will lie in a well-known Hilbert space (i.e., the space of square-summable sequences, denoted $\ell_2(-\infty, \infty)$). The information set of agent i at time t, denoted by Ω_t^i, represents the current and past values of the variables observed by trader i. There will be two types of information available to each agent – exogenous and endogenous. Exogenous information, denoted by U_t^i, is by definition not affected by market forces and is assumed to covariance stationary (see restrictions (R1)–(R2) below). Endogenous information is generated through market interactions of differentially informed agents.

Following Futia [15], covariance stationary equilibria require the following restrictions on the time evolution of the information set of agent i.

Definition 1. *The information set of agent i, Ω_t^i, is said to be stationary if the following two properties are satisfied,*

$$\Omega_t^i \subseteq \Omega_{t+1}^i, \qquad \forall\, t, i \tag{R1}$$

$$L\Omega_{t+1}^i = \Omega_t^i \qquad \forall\, t, i. \tag{R2}$$

The first condition requires that agents learn over time and do not forget information. The second condition ensures that the information process evolves according to a stationary process and follows from the following theorem due to Kolmogorov [22].

Theorem 1. *A stochastic process $\{X_t\}$ in Hilbert space H is stationary, if and only if there exists a unique unitary operator L on its time domain and an $X \in H(X)$ such that*

$$X_t = L^t X, \quad t \in \mathbb{Z}.$$

Proof. See [22]. □

In what follows, the information set of trader i is generated by *square-summable* sequences of stationary processes–[15] referred to random variables with this property as admissable. Therefore, we need the following corollary to Theorem 1, which states that any linear transformation of a stationary process with constant coefficients is also stationary.

Corollary 1. *Let X_t be a stationary process in H with the lag operator L, then the stochastic process $\{Y_t; t \in \mathbb{Z}\}$ defined by*

$$Y_t = \lim_{n\to\infty} \sum_{k=-n}^{n} a_k X_{t-k}, \tag{5}$$

is also stationary with the same lag operator L.

Proof. See [31]. □

Given that the economic shocks $\{u_t, \varepsilon_{1t}, \varepsilon_{2t}\}$ are assumed to be stationary processes and that trader i's information set is comprised of square-summable sequences of the economic shocks, (R2) will be satisfied. Couple this with the assumption that agents behave rationally and (R1)–(R2) will always hold in the analysis below.

Remark 1. *Given the restrictions on the time evolution of agent i's information set (* (R1)–(R2)*), the Projection Theorem (see, [12]) implies that agent i's conditional expectation of* p_{t+1} *is the orthogonal projection of* p_{t+1} *on the smallest closed subspace which contains* Ω_t^i*. This subspace will include both exogenous and endogenous information, and since the collection of variables is jointly normal, conditional expectations will reduce to linear least-squares projections. Let* $H_x(t)$ *denote the space spanned by square-summable linear combinations of current and past values of* x*. Then trader i's expectation of* p_{t+1} *is given by*

$$\mathbb{E}_t^i(p_{t+1}) = \Pi[p_{t+1}|\Omega_t^i] = \Pi[p_{t+1}|U_t^i \bigvee H_p(t)] \tag{6}$$

where Π *denotes linear least-squares projection and* $U_t^i \bigvee H_p(t)$ *is standard notation for the "linear space spanned by* U_t^i *and* $H_p(t)$*."*

Agents use all exogenous information *and* information generated by current and past values of the equilibrium price in evaluating the expectation of tomorrow's price. The expectation of investor i is then simply the orthogonal projection of p_{t+1} onto the subspace generated by U_t^i and $H_p(t)$. Let Ξ_t denote the set of information known by all traders at date t (i.e., $\Xi_t \equiv \bigcup_i \Omega_t^i$). We are now ready to define a rational expectations equilibrium (REE).

Definition 2. *A rational expectations equilibrium is a stochastic process for* $\{p_t\}$*, with* $p_t \in \Xi_t$*, that satisfies market clearing* (4)*, where expectations are formed according to* (6)*.*

The requirement, $p_t \in \Xi_t$, is what Futia [15] referred to as the axiom of "no divine revelation." This condition prevents equilibrium prices at date t from conveying any more information than that which could be in principle be available to traders at date t. In a symmetric equilibrium, agents' expectations coincide:

Definition 3. *We say a REE is symmetric if after observing the history of equilibrium prices* $\{p_{t-j}\}_{j=0}^{\infty}$ *all traders have identical information and make the same forecasts. That is*

$$\mathbb{E}_t^i(p_{t+1}) = \Pi[p_{t+1}|U_t^i \bigvee H_p(t)] = \mathbb{E}_t^j(p_{t+1}) = \Pi[p_{t+1}|U_t^j \bigvee H_p(t)]$$

for all i *and* j*.*

3.2 Solution Procedure

This section outlines a solution procedure that highlights the contributions of Sargent [34], Kasa [19], and Pearlman and Sargent [29], but also adds conditions that guarantee equilibrium even when operating outside of the specialized setup of Townsend [36].

The equilibrium of the model is computed in the following four steps. First, each trader uses all available information at time t (Ω_t^i) to form beliefs about the current price process. Second, the conditional expectation of p_{t+1} will be taken via Wiener-Kolmogorov optimal prediction formulas. Third, the appropriate form of (4) is then used to impose market clearing. In solving the equilibrium fixed-point problem, I follow Kasa [19] in appealing to the Riesz-Fischer Theorem and deriving the solution in the frequency domain.[8] Recall that the current information set of investor i includes the current price and past prices. Surely traders will condition on past prices; therefore, the fourth step is to ensure no additional information can be gleaned from the current price than that which was assumed at step one.

It is the fourth and final step that must be satisfied for an equilibrium to exist. I will refer to the "equilibrium" generated by steps one through three as a *candidate* equilibrium. This is to emphasize that agents of the economy can extract information from endogenous variables–in this case, the price of the asset. Hence in order for the candidate equilibrium to be an actual equilibrium, it must be the case that the price of the asset does not reveal any additional information than that assumed at step one. If the candidate equilibrium price does reveal information to agents that they did not have in forming expectations, then the solution procedure will be repeated with the updated information sets of the traders. In a sense, we must have an"informational fixed point" in order for the price process to be sustainable in equilibrium. Unlike Pearlman and Sargent [29], who employ invariant subspace methods of Pearlman et al. [28] to keep track of each agent's forecast error, I argue that frequency domain techniques make it easy to keep track of the Hilbert space generated by endogenous variables. The assumption here is that agents have access to vector autoregression (VAR) technology, and the information content of the asset price will be shown to hinge upon a simple invertibility condition.

It is important at this stage to explain in greater detail why it is more efficient to solve models of higher-order beliefs in the frequency domain. As described in Kasa [19], the benefit of working in the frequency domain is pure computational convenience. Working in the time domain, Sargent [34] shows how to convert the infinite dimensional state space associated with the forecasting the forecasts of others problem of Townsend [36] into a finite dimensional system. Rather than guessing an *infinite*-order autoregressive representation for beliefs (the state variable), Sargent models agents as forecasting by fitting low-order autoregressive, moving-average (ARMA) representations. By introducing moving-average components in agents' perceptions, Sargent utilizes the ability of low-order ARMA representations to replicate the space of some infinite-order AR representations. The drawback of this approach is that not only does one have to solve a fixed point problem in the coefficients of the ARMA process, but the *order* of the ARMA process must be matched as well. This led to the insight of Kasa, who shows how this two-step process can be condensed into a single step by working in the frequency domain. As opposed to guessing a functional form for beliefs, applying the Kalman filter, and then attempting to match coefficients, the frequency domain allows one to work with a functional fixed point problem. Therefore coefficients and order of the ARMA process are matched simultaneously by using the theory of the residue

[8] Recall the Riesz-Fischer Theorem states there is an equivalence (i.e., an isometric isomorphism) between the space of square-summable sequences denoted by $\ell_2(-\infty, \infty)$ and the space of square integrable functions, $L^2[\pi, -\pi]$; the former is referred to as the time domain and the latter the frequency domain.

calculus.

As a purely expositional example of how the frequency domain improves computational efficiency, suppose investor i believes the asset price follows the process

$$p_t = K(L)\varepsilon_t,$$

where $K(L)$ is a polynomial in nonnegative powers of the lag operator L with square-summable coefficients. Working in the time domain, one has to take an explicit stance on the functional form of $K(L)$ (i.e., ARMA(1,1), ARMA(2,1), etc.). After an optimization routine (typically involving an application of the Kalman filter), one has to match the coefficients of the process in the usual way. However in equilibrium, one must also match the *order* of the process. For example, investors who guess that the price process $K(L)$ follows an ARMA(1,1) might be able to reduce their forecast errors by changing their guess to an ARMA(2,3).[9] Conversely in the frequency domain, one does not have to take an explicit stance on the function form of $K(L)$ in forming investors' beliefs. The only assumption is that $K(L)$ has square-summable coefficients. Thus in the time domain, one must solve the fixed point problem in the coefficients and order of the process separately. In the frequency domain, these are solved simultaneously.

3.3 Homogenous Information

As a baseline model, it is useful to first assume a symmetric information structure that avoids infinite regress.[10] To this end, suppose that every investor observes past prices, the coupon stream, and net supplies. That is, the common information set of every trader i is given by

$$\Omega_t^i = \{p_{t-j}, c_{t-j}, s_{t-j} : j \geq 0\} \qquad \forall\ i.$$

Given this informational assumption, an investor's belief about the average equilibrium price will be represented by a linear combination of current and past values of $\{p_t, c_t, s_t\}$. In the absence of sunspots, these stochastic processes will be driven by the underlying shocks $\{u_t, \varepsilon_{1t}, \varepsilon_{2t}\}$. For reasons outlined in Whiteman [41], it is much simpler to calculate equilibrium prices and quantities if agents' expectations are computed under the assumption that they can see these underlying shocks. This "candidate" equilibrium will be realizable if and only if the space spanned by current and past values of the candidate processes $\{c_t, s_t, p_t\}$ is identical to that spanned by current and past values of the underlying shocks $\{u_t, \varepsilon_{1t}, \varepsilon_{2t}\}$. Therefore, trader i's expectation of p_{t+1} is assumed to be given by

$$\mathbb{E}_t^i(p_{t+1}) = \Pi[p_{t+1}|H_s(t) \bigvee H_p(t) \bigvee H_c(t)] = \Pi[p_{t+1}|H_u(t) \bigvee H_{\varepsilon_1}(t) \bigvee H_{\varepsilon_2}(t)]. \qquad (7)$$

Thus we proceed by assuming agents see the economic shocks, compute a candidate equilibrium under that assumption, and then check to verify that the candidate equilibrium process

[9] [34] referred to an equilibrium price process in which agents have no incentive to guess an alternative functional form as "full order."

[10] This case will not only serve as a benchmark but the next section shows how *ex ante* disparate information structures can degenerate to this homogeneous information case.

$\{c_t, s_t, p_t\}$ would enable agents to uncover $\{u_t, \varepsilon_{1t}, \varepsilon_{2t}\}$. If this recovery is possible, then the corresponding equilibrium is a rational expectations equilibrium (REE).

Thus, step one of the solution process suggests we begin with the tentative assumption that every trader believes the average equilibrium price to be given by

$$p_t = \sum_{j=0}^{\infty} G_j \; u_{t-j} + \sum_{j=0}^{\infty} D_j \; \varepsilon_{1,t-j} + \sum_{j=0}^{\infty} F_j \; \varepsilon_{2,t-j} \equiv G(L)u_t + D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t} \tag{8}$$

where $G(L)$, $D(L)$, and $F(L)$ are (possibly) infinite-order square summable polynomials in the lag operator L.[11] This is an assumption because it is not clear that agents will be able to observe the underlying shocks of the economy, especially when they are differentially informed. Hence, the fourth step of the solution procedure must verify the guess made by (8).

The second step in the solution method is to calculate the conditional expectations using the Wiener-Kolmogorov optimal prediction formula,[12]

$$\begin{gathered}
\mathbb{E}_t^i(p_{t+1}) = L^{-1}[G(L) - G_0]u_t + L^{-1}[D(L) - D_0]\varepsilon_{1t} + L^{-1}[F(L) - F_0]\varepsilon_{2t} \\
\mathbb{E}_t(c_{t+1}) = L^{-1}[C(L) - C_0]u_t.
\end{gathered}$$

Imposing market clearing (4) and rearranging, one obtains

$$\begin{gathered}
\varphi p_t = \int_0^1 \mathbb{E}_t(p_{t+1} + c_{t+1})di - s_t \\
\varphi[G(L)u_t + D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t}] = L^{-1}[G(L) - G_0]u_t + L^{-1}[D(L) - D_0]\varepsilon_{1t} \\
+L^{-1}[F(L) - F_0]\varepsilon_{2t} + L^{-1}[C(L) - C_0]u_t - [A(L)\varepsilon_{1t} + B(L)\varepsilon_{2t}].
\end{gathered}$$

The third step of the solution process is to solve the corresponding fixed point problem in the frequency domain. As mentioned above, this is simply for computational convenience. Assuming that this expression holds for all realizations of u_t, ε_{1t} and ε_{2t}, the coefficients on u_s, ε_{1s} and ε_{2s} must match for every s. In lieu of solving this infinite sequential problem, one can solve an equivalent functional problem by examining the corresponding power series equalities

$$\begin{gathered}
\varphi G(z) = z^{-1}[G(z) - G_0] + z^{-1}[C(z) - C_0] \\
\varphi D(z) = z^{-1}[D(z) - D_0] - A(z), \\
\varphi F(z) = z^{-1}[F(z) - F_0] - B(z).
\end{gathered}$$

Finding the appropriate functions $G(z)$, $D(z)$, and $F(z)$ in the frequency domain yields the following proposition.

[11] It is important to note that by working in the frequency domain one does not have to take a stance on the explicit functional form of beliefs, $G(L)$, $D(L)$ and $F(L)$. This is especially convenient when beliefs contain moving average components because MA representations are difficult to handle in the time domain.

[12] Note that we can write the demeaned coupon process as $c_t = C(L)u_t$ where $C(L) = (1 - \psi L)^{-1}$.

Proposition 1. *The candidate equilibrium price is unique and given by*

$$p_t = \left[\frac{LA(L) - \varphi^{-1}A(\varphi^{-1})}{1 - \varphi L}\right]\varepsilon_{1t} + \left[\frac{LB(L) - \varphi^{-1}B(\varphi^{-1})}{1 - \varphi L}\right]\varepsilon_{2t} + \left[\frac{C(\varphi^{-1}) - C(L)}{1 - \varphi L}\right]u_t, \quad (9)$$

Proof. Due to the symmetry of the problem and the assumption that the shocks are not correlated, we can focus on solving the fixed-point problem for one process, say ε_{1t}. A little algebra gives

$$D(z)(1 - \varphi z) = D_0 + zA(z).$$

As mentioned above, solutions to the model will be sought in the space spanned by square summable linear combinations of the underlying fundamental shocks, and thus it must be the case that the coefficients D_j are square summable. The requirement of square-summability in the time domain corresponds to the requirement that $D(z)$ be analytic on the open unit disk $|z| < 1$ in the frequency domain. Given $\varphi > 1$, this function will not be analytic unless the free parameter D_0 removes the singularity at $z = \varphi^{-1}$. This is achieved by setting the residue equal to zero and solving for D_0, which yields

$$\lim_{z \to \varphi^{-1}} D(z)(1 - \varphi z) = D_0 + \varphi^{-1}A(\varphi^{-1}) = 0$$
$$D_0 = -\varphi^{-1}A(\varphi^{-1}).$$

This implies $D(z)$ is *unique* and given by

$$D(z)^* = \frac{zA(z) - \varphi^{-1}A(\varphi^{-1})}{(1 - \varphi z)}. \quad (10)$$

□

The candidate equilibrium price involves three instances of the Hansen and Sargent [16] prediction formula, and notice that (10) and (9) specify the candidate equilibrium as a function of the fundamental processes of the model. These cross-equation restrictions are what Sargent [32] and many others have referred to as "the hallmark of rational expectations models."

The fourth and final step in the solution procedure is to determine whether the *ex ante* informational assumptions support such a price process. Recall that (8) assumes investors are able to observe the fundamental shocks to the economy. We can now prove (or disprove) this conjecture by constructing the observer system for agent i, which includes all of the conditioning variables on the left hand side and fundamental shocks on the right hand side. Setting this system up for the symmetric case one obtains,

$$\begin{bmatrix} c_t \\ s_t \\ p_t \end{bmatrix} = \begin{bmatrix} C(L) & 0 & 0 \\ 0 & A(L) & B(L) \\ \frac{C(\varphi^{-1}) - C(L)}{1-\varphi L} & \frac{LA(L) - \varphi^{-1}A(\varphi^{-1})}{1-\varphi L} & \frac{LB(L) - \varphi^{-1}B(\varphi^{-1})}{1-\varphi L} \end{bmatrix} \begin{bmatrix} u_t \\ \varepsilon_{1t} \\ \varepsilon_{2t} \end{bmatrix}$$

or more compactly

$$\mathbf{y}_t = \mathcal{M}(L)\boldsymbol{\epsilon}_t. \tag{11}$$

If this system is invertible in nonnegative powers of L, then $\boldsymbol{\epsilon}_t$ may be obtained as a square-summable linear combination of current and past $\mathbf{y}_t$. This would imply that the guess made by (8) is verified and that (11) is a REE. This suggests that if agents are equipped with basic multivariate statistical analysis (e.g., VAR analysis), then knowledge of past and present $\mathbf{y}$ is equivalent to knowledge of past and present $\boldsymbol{\epsilon}$. More formally, the Hilbert spaces generated by $\{\mathbf{y}_t, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, ...\}$ and $\{\boldsymbol{\epsilon}_t, \boldsymbol{\epsilon}_{t-1}, \boldsymbol{\epsilon}_{t-2}, ...\}$ must be identical (in mean-square sense).

In order to proceed, we need to place more structure on the model; anticipating analysis below, we follow Singleton in the specification:[13]

Assumption 1. *The supply and coupon processes are given by:*

$$C(L) = \frac{1}{(1-\psi L)}, \qquad A(L) = \frac{1}{(1-\rho L)}, \qquad B(L) = 1 + \varsigma L, \qquad 0 \le |\psi|, |\rho|, |\varsigma| \le 1$$

Note that under this assumption, supply is the sum of a first-order autoregression (AR(1)) and a first-order moving average (MA(1)). Depending on the relative sizes of $\sigma^2_{\varepsilon_1}$ and $\sigma^2_{\varepsilon_2}$ and whether ρ and ς are nonzero, this is general enough to include the following special cases for the univariate representation of supply: white noise, MA(1), AR(1) (in the limit as $\sigma^2_{\varepsilon_2}/\sigma^2_{\varepsilon_1} \to \infty$), ARMA(1,1) and ARMA(2,1). It is now possible to determine conditions for the invertibility of (11).

Lemma 1. *The vector moving average representation* (11) *is invertible (making* (9) *a REE) provided*

$$\left| \frac{\alpha\varsigma\rho}{\alpha(\rho-\varsigma)+\varsigma\rho} \right| \le 1. \tag{12}$$

Proof. A necessary and sufficient condition for (11) to be a fundamental (Wold) representation for $[c_t, s_t, p_t]'$ and thus invertible, is that the determinant of $\mathcal{M}(z)$ be analytic and have no zeros inside the open unit disk. By direct calculation,

$$\begin{aligned} \det \mathcal{M}(z) &= C(z)\left[\frac{A(z)[zB(z)-\alpha^{-1}B(\alpha^{-1})]}{(1-\alpha z)} - \frac{B(z)[zA(z)-\alpha^{-1}A(\alpha^{-1})]}{(1-\alpha z)}\right] = 0 \\ &= \frac{\alpha^{-2}[\alpha(\rho-\varsigma)+\varsigma\rho+\alpha\varsigma\rho z]}{(\alpha-\rho)(1-\rho z)}; \end{aligned} \tag{13}$$

the stated condition guarantees that det $\mathcal{M}(z)$ does not contain any zeros inside the unit circle, and therefore (11) is invertible.[14] □

Invertibility makes (11) a fundamental (Wold) representation. This is important because it implies $\mathcal{M}(L)$ has a one-sided inverse in nonnegative powers of L, so a corollary of the

[13] As in [35], the model was also solved with $A(L) = 1 + \phi L$. Adopting a different representation for $A(L)$ will alter the solution of the model slightly but the main results found here continue to hold given $|\phi| < 1$.

[14] Notice that if $\varsigma = 0$, representation (11) will always be invertible.

above lemma is that $H_c(t) \bigvee H_s(t) \bigvee H_p(t) \equiv H_u(t) \bigvee H_{\varepsilon_1}(t) \bigvee H_{\varepsilon_2}(t)$. Thus, the observables c_t, s_t and p_t span the same linear space as the underlying fundamental shocks u_t, ε_{1t} and ε_{2t}, and therefore the Hilbert spaces generated by $\{\mathbf{y}_t, \mathbf{y}_{t-1}, \mathbf{y}_{t-2}, ...\}$ and $\{\boldsymbol{\epsilon}_t, \boldsymbol{\epsilon}_{t-1}, \boldsymbol{\epsilon}_{t-2}, ...\}$ are identical (in mean-square sense). Thus, the equality

$$\mathbb{E}_t^i(p_{t+1}) = \Pi[p_{t+1}|H_c(t) \bigvee H_p(t) \bigvee H_s(t)] = \Pi[p_{t+1}|H_u(t) \bigvee H_{\varepsilon_1}(t) \bigvee H_{\varepsilon_2}(t)] \tag{14}$$

holds for all i. By allowing traders to guess an equilibrium price that is a linear combination of the underlying shocks (8), equality (14) was implicitly assumed to hold. This suggests, and was subsequently proven by Lemma 1, that by observing the combination of the history of the coupon process, net supplies and equilibrium prices and having knowledge of VAR analysis, agents would be able to infer the underlying shocks. Therefore, the guess given by (8) assumed the shocks $\{u_t, \varepsilon_{1t}, \varepsilon_{2t}\}$ were in the information set of agent i. In order to prove this claim, the observable equilibrium variables $\{c_t, s_t, p_t\}$ had to uncover $\{u_t, \varepsilon_{1t}, \varepsilon_{2t}\}$. Of course this relationship does not have to hold in equilibrium. The next section studies whether disparate expectations are preserved in this setup under Singleton's information structure.

3.4 Asymmetric Information

Following the framework of the U.S. bond market, in which there is little uncertainty concerning the coupon stream, Singleton introduced asymmetric information via the supply process.[15] Suppose there are two distinct groups of traders, in proportion k and $(1-k)$. Traders are not able to observe net supply directly, but *every* trader receives a private, noisy signal on $A(L)\varepsilon_{1t}$ and a public signal on $B(L)\varepsilon_{2t}$.[16] Denote the two signals by

$$v_{it} = A(L)\varepsilon_{1t} + \eta_{it}, \qquad v_t = B(L)\varepsilon_{2t}$$

where η_{it} is assumed to be i.i.d, normally distributed with finite variance and is uncorrelated with all other shocks. Not only will each individual trader receive different realizations of η, but the two groups of traders differ in their qualities of information. Traders in group 1 (in proportion k) see private signals with smaller variance ($\sigma^2_{\eta 1} < \sigma^2_{\eta 2}$). Of course agents will also be able to condition on current and past prices, and the information set of agent i at time t is given by

$$\Omega_t^i = \{c_{t-j}, v_{t-j}, v_{it-j}, p_{t-j} : j \geq 0\}. \tag{15}$$

The central question is, will this heterogenous information be enough to generate and preserve disparate expectations in equilibrium? The exogenous information structure is asymmetric; each investor receives a different realization of η_t and the two groups receive different

[15] It is important to note that the main results of the paper continue to hold if asymmetric information is introduced via the dividend stream.

[16] Assuming traders observe a public signal does *not* imply that traders observe contemporaneous realizations of the shock ε_{2t}. The assumption of a public signal on ε_2 is driven by the Singleton-Townsend truncation technique, which assumes all traders observe ε_2 with a two-period lag. The results of this section would continue to hold even if $B(L) = L^n$, where n is a finite integer.

qualities of information on average. However, in equilibrium, each trader will also extract information from current and past prices. If the equilibrium price provides a rich enough information structure to bridge the gap among traders, then the equilibrium will degenerate into the one studied in Section 3.3.

The difference between this section and the previous section is that the agents must first solve the signal extraction problem, which relates the signals to the underlying shocks. After solving this problem, agents will then use this information to generate a guess of the equilibrium price process. Consider the following signal system for agent i,

$$\begin{bmatrix} v_{it} \\ v_t \end{bmatrix} = \begin{bmatrix} A(L) & 1 & 0 \\ 0 & 0 & B(L) \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \eta_{it} \\ \varepsilon_{2t} \end{bmatrix}$$

or more compactly,

$$\mathbf{v}_t = \mathcal{H}(L)\boldsymbol{\nu}_t. \tag{16}$$

Since there are more shocks than observables, agent i will not be able to "see" or infer both economically fundamental shocks $(\varepsilon_{1t}, \varepsilon_{2t})$. This implies, *a priori*, that a relationship analogous to (14) will not hold. That is,

$$E_t^i(p_{t+1}) = \Pi[p_{t+1}|H_c(t) \bigvee H_{v_i}(t) \bigvee H_v(t)] \neq \Pi[p_{t+1}|H_u(t) \bigvee H_{\varepsilon_1}(t) \bigvee H_{\varepsilon_2}(t)]. \tag{17}$$

It is useful at this stage to distinguish the signal extraction problem solved here with the one solved when implementing the truncation technique of Townsend [36] and Singleton [35]. The truncation technique assumes traders observe the fundamental shock ε_1 with a two-period lag,

$$\begin{bmatrix} \varepsilon_{1,t-2} \\ v_{it} \\ v_t \end{bmatrix} = \begin{bmatrix} L^2 & 0 & 0 \\ A(L) & 1 & 0 \\ 0 & 0 & B(L) \end{bmatrix} \begin{bmatrix} \varepsilon_{1t} \\ \eta_{it} \\ \varepsilon_{2t} \end{bmatrix}$$
$$\mathbf{x}_t = \mathcal{N}(L)\boldsymbol{\epsilon}_t \tag{18}$$

The above representation is not a fundamental one due to the zero in the determinant of $\mathcal{N}(L)$ at $L = 0$; therefore traders will not be able to infer the economically fundamental shocks ε_{1t} and $\varepsilon_{1,t-1}$.[17] Comparing the informational assumption of (18) with that of (16), representation (16) implies *less* information in that investors are not able to see ε_{1t} with a two-period lag. In fact, the only initial information given to traders concerning ε_{1t} is that which can be extracted from the signal v_{it}. By solving the model analytically without relying on truncating the state space, we are able to assess the size of the approximation error when invoking truncation. Foreshadowing results to come, the conclusions reached here suggest that truncation may poorly approximate the actual equilibrium.

The important question then becomes, given representation (16), what will the agents

[17] Interestingly, even without implementing the truncation technique, agents of Townsend's economy solve a signal extraction problem of this form (see, [19] Eq. (23)).

be able to infer? To give a concrete example, I adopt Assumption 1 and seek a fundamental representation to replace (16). The fundamental representation will then be used by the traders to form a guess of the equilibrium price. The following proposition finds the unique fundamental representation to replace (16)

Lemma 2. *The fundamental signal system for agent i in group k is given by*

$$\begin{bmatrix} v_{it} \\ v_t \end{bmatrix} = \begin{bmatrix} \mathcal{J}_1(L) & 0 \\ 0 & B(L) \end{bmatrix} \begin{bmatrix} \xi_{it} \\ \varepsilon_{2t} \end{bmatrix} \tag{19}$$

where $\mathcal{J}_1(L) = \frac{1-\lambda_1 L}{1-\rho L}$

Proof. We seek a fundamental (Wold) representation for the private signal (v_{it})– call it $\mathcal{J}_1(L)\xi_{it}$. This is accomplished by factorization of the covariance generating function of the private signal, which is given by[18]

$$g_{v_i}(z) = \frac{\sigma^2_{\varepsilon_1}}{(1-\rho z)(1-\rho z^{-1})} + \sigma^2_{\eta_1}.$$

In establishing a fundamental representation, we seek a λ_1 and $\sigma^2_{\xi_1}$ such that

$$\sigma^2_{\varepsilon_1} + \sigma^2_{\eta_1}(1-\rho z)(1-\rho z^{-1}) = \sigma^2_{\xi_1}(1-\lambda_1 z)(1-\lambda_1 z^{-1}).$$

Setting λ_1 equal to the smaller root ensures $|\lambda_1| < 1$ and yields[19]

$$\lambda_1 = \frac{1}{2}\left[\left(\frac{\sigma^2_{\varepsilon_1}}{\sigma^2_{\eta_1}\rho}\right) + \left(\frac{1}{\rho} + \rho\right) - \left\{\left[\frac{\sigma^2_{\varepsilon_1}}{\sigma^2_{\eta_1}\rho} + \left(\frac{1}{\rho} + \rho\right)\right]^2 - 4\right\}^{1/2}\right].$$

The variance $\sigma^2_{\xi_1}$ is then found by the formula

$$\sigma^2_{\xi_1} = \frac{g_z(1)}{\mathcal{J}_1(1)^2} = \frac{\sigma^2_{\varepsilon_1} + \sigma^2_{\eta_1}(1-\rho)^2}{(1-\lambda_1)^2}.$$

□

Note that Type 2 investors will have an analogous fundamental representation $\mathcal{J}_2(L)\xi_{it}$ with λ_2 and $\sigma^2_{\xi_2}$ each a function of $\sigma^2_{\eta_2}$.

This factorization puts the signal in a form that the agents can use to predict next period's price and it also tells us the relationship between the signals and the fundamental shocks. By construction, (19) a fundamental (Wold) representation and can now be used to generate trader i's guess of the equilibrium price.

[18] See [42] or [33] Chapter XI Section 18.

[19] This also ensures ξ_{it} lies in the linear space spanned by current and lagged v_{it}'s. In other words, that ξ_{it} is the one step ahead prediction error of predicting v_{it} from its own past, $\xi_{it} = \Pi(v_{it}|v_{it-1}, v_{it-2}, ...)$; and therefore the Hilbert spaces generated by $\{v_{it}, v_{it-1}, ...\}$ and $\{\xi_{it}, \xi_{it-1}, ...\}$ are equivalent.

Corollary 2. *Trader i's belief of the current price is given by*

$$p_t = D_i(L)\xi_{it} + F(L)\varepsilon_{2t} + G(L)u_t \tag{20}$$

where ξ_{it} *is related to the underlying shock* ε_{1t} *by the equation*

$$\xi_{it} = \mathcal{J}_1(L)^{-1}A(L)\varepsilon_{1t} + \mathcal{J}_1(L)^{-1}\eta_{it} = \frac{\varepsilon_{1t}}{1-\lambda_1 L} + \frac{1-\rho L}{1-\lambda_1 L}\eta_{it}. \tag{21}$$

Note that as the signal-to-noise ratio ($\sigma_{\varepsilon_1}/\sigma_{\eta_j}$) approaches infinity, λ_j approaches zero and the first component of (21) approaches ε_{1t}. Conversely as the signal-to-noise ratio approaches zero, λ_j approaches ρ. Therefore traders in group 1, who receive a more precise signal, will have a more accurate guess of the price process on average. Further, because the entire price sequence is observable in equilibrium, in order that agents have perceptions about the serial correlation properties of prices that are consistent with what is observed, the $D_i(L)$ functions must be identical within groups and proportional across groups:

$$\text{for } i \le k: \quad D_i(L) = \chi_1 D(L), \qquad \text{for } k \le i: \quad D_i(L) = \chi_2 D(L).$$

Still, this information setup implies disparate expectations for *every* i

$$\mathbb{E}_t^i(p_{t+1}) = \Pi[p_{t+1}|H_c(t)\bigvee H_{v_i}(t)\bigvee H_v(t)] = \Pi[p_{t+1}|H_u(t)\bigvee H_{\xi_i}(t)\bigvee H_{\varepsilon_2}(t)].$$

In particular, for $i \le k$,

$$\mathbb{E}_t^i p_{t+1} = L^{-1}[D(L)-D_0]\chi_1\xi_{it} + L^{-1}[F(L)-F_0]\varepsilon_{2t} + L^{-1}[G(L)-G_0]u_t.$$

Given that the η_{it}'s are i.i.d., I will assume that a version of the strong law of large numbers holds for each of the two groups so that the overall impact of the idiosyncratic shocks averages to zero. That is,

$$E_t^1 p_{t+1} \equiv \int_0^k \mathbb{E}_t^i p_{t+1} di = kL^{-1}[D(L)-D_0]\xi_{1t} + kL^{-1}[F(L)-F_0]\varepsilon_{2t} + kL^{-1}[G(L)-G_0]u_t$$

where

$$\xi_{1t} = \chi_1 \int_0^k \xi_{it} di = \frac{\chi_1\varepsilon_{1t}}{(1-\lambda_1 L)}$$

and analogously for $k \le i \le 1$. This assumption will *not* by itself lead to a symmetric REE because the traders have different qualities of information ($\sigma_{\eta_1}^2 < \sigma_{\eta_2}^2$) which implies $\xi_1 \ne \xi_2$. It is important to note that $\chi_1 > \chi_2$ and $\lambda_1 < \lambda_2$ implies traders of group 1 have more accurate perceptions and therefore smaller one-step-ahead forecast errors. Thus heterogenous information and disparate expectations now exists across the two *groups*,

$$\mathbb{E}_t^1(p_{t+1}) = \Pi[p_{t+1}|H_{\xi_1}(t)\bigvee H_{\varepsilon_2}(t)\bigvee H_u(t)] \ne \mathbb{E}_t^2(p_{t+1}) = \Pi[p_{t+1}|H_{\xi_2}(t)\bigvee H_{\varepsilon_2}(t)\bigvee H_u(t)]. \tag{22}$$

Because the idiosyncratic information integrates out, the equilibrium price will be a linear function of ε_{1t}. Thus, $p_t = D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t} + G(L)u_t$, and χ_1 and χ_2 solve

$$\sigma^2_{\varepsilon_1} = \chi_1^2\sigma^2_{\xi_1} = \chi_2^2\sigma^2_{\xi_2}$$

i.e.,

$$\chi_1 = \left(\frac{\sigma^2_{\varepsilon_1}(1-\lambda_1)^2}{\sigma^2_{\varepsilon_1} + \sigma^2_{\eta_1}(1-\rho)^2}\right)^{1/2} \quad \chi_2 = \left(\frac{\sigma^2_{\varepsilon_1}(1-\lambda_2)^2}{\sigma^2_{\varepsilon_1} + \sigma^2_{\eta_2}(1-\rho)^2}\right)^{1/2}.$$

It is straightforward to verify that if p_t is of the form $D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t} + G(L)u_t$, then the average forecast errors $p_{t+1} - \mathbb{E}^1_t p_{t+1}$ and $p_{t+1} - \mathbb{E}^2_t p_{t+1}$ are serially correlated. More troubling is the fact that individual forecast errors are also serially correlated. If this serial correlation can be exploited to improve predictions (as will be shown), perceptions (e.g., (20)) will not match reality, and we will not have a REE. The extent to which this additional information generated by the candidate price process can be exploited is the crux of the issue. Letting i and j denote the two groups of traders, the market clearing condition is

$$\varphi p_t = \int_0^k \mathbb{E}^i_t(p_{t+1} + c_{t+1})di + \int_k^1 \mathbb{E}^j_t(p_{t+1} + c_{t+1})dj - s_t \tag{23}$$

$$\varphi[D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t} + G(L)u_t] = kL^{-1}[D(L) - D_0]\frac{\varepsilon_{1t}\chi_1}{1-\lambda_1 L} + (1-k)L^{-1}[D(L) - D_0]\frac{\varepsilon_{1t}\chi_2}{1-\lambda_2 L}$$
$$+L^{-1}[F(L) - F_0]\varepsilon_{2t} + L^{-1}[G(L) - G_0]u_t + L^{-1}[C(L) - C_0]u_t - A(L)\varepsilon_{1t} - B(L)\varepsilon_{2t}.$$

It is easy to see that the solution for $F(z)$ and $G(z)$ will be the same as the symmetric case; specifically,

$$F(z)^* = \frac{zB(z) - \varphi^{-1}B(\varphi^{-1})}{1-\varphi z}$$
$$G(z)^* = \frac{C(\varphi^{-1}) - C(z)}{1-\varphi z}.$$

The third step of the solution procedure is to equate coefficients on $\varepsilon_{1t}, \varepsilon_{1t-1}, \ldots$, which yields the power series equality

$$D(z)[-\lambda_1\lambda_2\varphi z^3 + (\lambda_2+\lambda_1)\varphi z^2 - ((1-k)\lambda_1\chi_2 + \lambda_2\chi_1 k + \varphi)z + k\chi_1 + (1-k)\chi_2]$$
$$= D_0[(1-\lambda_2 z)k\chi_1 + (1-\lambda_1 z)(1-k)\chi_2] + z(1-\lambda_1 z)(1-\lambda_2 z)A(z).$$

The roots of the cubic equation

$$-\lambda_1\lambda_2\varphi z^3 + (\lambda_2+\lambda_1)\varphi z^2 - (\lambda_2\chi_1 k + (1-k)\lambda_1\chi_2 + \varphi)z + k\chi_1 + (1-k)\chi_2 \tag{24}$$

will determine the uniqueness of the candidate equilibrium encountered. The following proposition shows that for the supply process given by Assumption 1 and assuming the sum of the gross interest rate and supply response exceeds unity ($\varphi = \xi + \alpha > 1$), there is a unique candidate price.

Proposition 2. *Given $\varphi > 1$ and Assumption 1, the price process generated by information* (15) *and the market clearing condition* (23) *is unique.*

Proof. See Appendix B. □

In order for the equilibrium price to be unique, we need *exactly* one root of (24) to lie inside the unit circle. If no root lies inside the unit circle, then the free parameter D_0 will not be pinned down and $D(z)$ will not be unique. If more than one root lies inside the unit circle, a square-summable $D(z)$ satisfying (24) does not exist. The proof found in Appendix B shows that there exists exactly one root that lies inside the unit circle. Let θ denote this root. Then D_0 will be set to remove this singularity as before; in this case we have

$$D_0^* = -\frac{\theta(1-\lambda_1\theta)(1-\lambda_2\theta)A(\theta)}{(1-\lambda_2\theta)k\chi_1 + (1-\lambda_1\theta)(1-k)\chi_2}.$$

Substituting D_0^* into $D(z)$

$$\begin{aligned} &D(z)[-\lambda_1\lambda_2\varphi z^3 + (\lambda_2+\lambda_1)\varphi z^2 - ((1-k)\lambda_1\chi_2 + \lambda_2\chi_1 k + \varphi)z + k\chi_1 + (1-k)\chi_2] \\ &= \frac{-(\theta(1-\lambda_1\theta)(1-\lambda_2\theta)A(\theta))[(1-\lambda_2 z)k\chi_1 + (1-\lambda_1 z)(1-k)\chi_2]}{((1-\lambda_2\theta)k\chi_1 + (1-\lambda_1\theta)(1-k)\chi_2)} + z(1-\lambda_1 z)(1-\lambda_2 z)A(z) \end{aligned} \tag{25}$$

or more compactly

$$D(L)^*\varepsilon_{1t} = \left[\frac{D_0^*[(1-\lambda_2 L)k\chi_1 + (1-\lambda_1 L)(1-k)\chi_2] + L(1-\lambda_1 L)(1-\lambda_2 L)A(L)}{-\lambda_1\lambda_2\varphi L^3 + (\lambda_2+\lambda_1)\varphi L^2 - ((1-k)\lambda_1\chi_2 + \lambda_2\chi_1 k + \varphi)L + k\chi_1 + (1-k)\chi_2}\right]\varepsilon_{1t}.$$

We now have our unique candidate equilibrium price process,

$$p_t = D(L)^*\varepsilon_{1t} + F(L)^*\varepsilon_{2t} + G(L)^*u_t \tag{26}$$

and we can construct the post-equilibrium observer system for a trader i:

$$\begin{bmatrix} c_t \\ v_{it} \\ v_t \\ p_t \end{bmatrix} = \begin{bmatrix} C(L) & 0 & 0 & 0 \\ 0 & A(L) & 0 & 1 \\ 0 & 0 & B(L) & 0 \\ G(L)^* & D(L)^* & F(L)^* & 0 \end{bmatrix} \begin{bmatrix} u_t \\ \varepsilon_{1t} \\ \varepsilon_{2t} \\ \eta_{it} \end{bmatrix}$$
$$\mathbf{v}_t' = \mathcal{H}(L)\boldsymbol{\epsilon}_t \tag{27}$$

Representation (27) corresponds to the information available to trader i at time t, Ω_t^i. If (27) is a fundamental (Wold) representation, then the Hilbert spaces spanned by $\{\mathbf{v}_t', \mathbf{v}_{t-1}', ...\}$ and $\{\boldsymbol{\epsilon}_t, \boldsymbol{\epsilon}_{t-1}, ...\}$ are equivalent, and every trader will be able to infer the shock ε_{1t} by observing the price sequence. In other words, even though we assumed (at (20)) that agents saw only a noisy signal on ε_{1t} when forming expectations, the market interactions of those agents injects enough information into the price to reveal the underlying ε_{1t} process. Thus after conditioning on the current price, every investor will have identical forecasts in equilibrium. This leads to the following proposition.

Proposition 3. *Under Assumption 1, the candidate price process* (26) *reveals the current and past realizations of the fundamental shocks* $\{\varepsilon_{1t}\}$, $\{\varepsilon_{2t}\}$ *and* $\{u_t\}$ *to all traders.*

Proof. See Appendix B. It is shown in Appendix B that the determinant of $\mathcal{H}(z)$ has no zeros inside the unit circle. Following Lemma 1, this proves that (27) is a fundamental representation and reveals u_t, ε_{1t} and ε_{2t}. □

A few observations are noteworthy at this point. First, the upshot is simply that the assumption of asymmetric information is not sustainable in equilibrium. Even the non-truncated signal extraction problem given by (16) and the additional layer of asymmetric information ($\sigma^2_{\eta_1} \neq \sigma^2_{\eta_2}$) provides too much information through market interactions to preserve asymmetric information. Traders will surely condition on past prices and update their forecasts accordingly. Showing (27) is a fundamental representation is tantamount to the argument that *all* traders will guess a price process of the form

$$p_t = D(L)\varepsilon_{1t} + F(L)\varepsilon_{2t} + G(L)u_t.$$

And therefore the idiosyncratic shock will not enter the average traders' perceptions. More importantly, the conditional expectation (17) will be replaced by

$$E^i_t(p_{t+1}) = \Pi[p_{t+1}|H_{\varepsilon_1}(t) \bigvee H_{\varepsilon_2}(t) \bigvee H_u(t)]. \tag{28}$$

All traders will share the same forecast! Hence there is no need to forecast the forecasts of others. This economy will then degenerate into the one studied in Section 3.3.

Second, this is a general result in the sense that it does not depend entirely on the law of large numbers and holds under weak assumptions. It has been known for some time–at least since Admati [2]–that without the assumption of two differentially informed *groups* of investors (recall $\sigma^2_{\eta 1} \neq \sigma^2_{\eta 2}$), invoking the strong law of large numbers makes the results found here trivial. But Proposition 3 proves that even the two disparately informed groups will share the same forecast in equilibrium. Also, Assumption 1 does not depend upon the degree of asymmetric information (i.e., does not depend upon $\sigma^2_{\eta 1}$ or $\sigma^2_{\eta 2}$) and is not that restrictive. For example, following [35] and assuming $\varsigma = 0$, implies the proposition holds for *any* stationary AR(1) process ($|\rho| \in (0, 1)$) regardless of the degree of asymmetric information.[20]

Finally, with an analytical solution in hand, it is easy to assess the approximation error associated with truncating the state space. The sustainable equilibrium price process will be identical to one derived in the previous section; specifically,

$$p^*_t = \left[\frac{LA(L) - \varphi^{-1}A(\varphi^{-1})}{1 - \varphi L}\right]\varepsilon_{1t} + \left[\frac{LB(L) - \varphi^{-1}B(\varphi^{-1})}{1 - \varphi L}\right]\varepsilon_{2t} + \left[\frac{C(\varphi^{-1}) - C(L)}{1 - \varphi L}\right]u_t. \tag{29}$$

[20] Moreover, it can be shown that these results extend to alternative time series specifications.

From the signal extraction problem given by (18), it is clear that the approximate price process derived from truncating the state space can be written as

$$p_t = p_t^* + \varrho_1 \varepsilon_{1t} + \varrho_2 \varepsilon_{1,t-1}. \tag{30}$$

where, as shown in Singleton, the coefficients ϱ_1 and ϱ_2 are functions of the degree of asymmetric information ($\sigma_{\eta_1}^2$ and $\sigma_{\eta_2}^2$).[21] Therefore as the degree of asymmetric information across groups becomes large, the approximate price process (30) deviates further away from the true equilibrium (29).

4 Conclusion

It seems doubtful that the bulk of fluctuations in asset markets is due primarily to differences in risk tolerance. A more likely alternative is that trade is generated by agents who are endowed with different sets of information. Unfortunately, tractable models of heterogenous information in a dynamic setting are difficult to construct and even more difficult to sustain in equilibrium.

This paper has demonstrated how endogenous variables reveal privately held information in a dynamic, asset pricing model. An explicit solution procedure that yielded analytical results was introduced. The solution procedure employed frequency domain techniques to keep track of information sets (Hilbert spaces) generated by endogenous variables. It was shown that a simple invertibility condition implied the revelation of economically fundamental shocks. Traders equipped with basic statistical analysis could then condition upon this "new" information and update their forecasts accordingly. Information structures previously believed to preserve disparate expectations were shown to be fully revealing.

One direction of future research would be to expand the solution technique to include a wider class of models.[22] For example, instead of examining models with simple point forecasts, one could imagine examining models with distribution forecasts. One could also expand the model by examining information generated by polynomials of the fundamental shocks, in lieu of simple linear combinations of the shocks. Assuming one stayed within the confines of a well-defined Hilbert space, the methodology established here could be used in these efforts.

Another line of research is to examine interesting ways to preserve higher-order beliefs. For example, Kasa et al. [20] preserve higher-order beliefs in a dynamic asset pricing model by assuming a special information structure that gives rise to zeros in the post-equilibrium observer equations. These zeros prohibit equilibrium prices from revealing the economically fundamental shocks. Alternatively, one could assume that agents are adaptively rational (e.g., [10]), have limited capacity (e.g., [43]), have access to short-lived information ([3], or act strategically to limit information dissemination (e.g., [8]). That these efforts seem to require very special circumstances to preserve differential information in equilibrium when disparate expectations seem so widespread, suggests that models of asymmetric information will constitute a fruitful area of research for some time to come.

[21] This can also be seen by finding the fundamental representation associated with (18).

[22] I thank an anonymous referee for suggesting this future line of research.

Appendix A: Equilibrium Multiplicity

Recall the following demand schedule for the risky asset

$$z_{it} = \frac{\mathbb{E}_t^i(p_{t+1} + c_{t+1}) - \alpha p_t}{\gamma \delta_i}$$

where $\delta_i = \text{var}_t^i(p_{t+1} + c_{t+1})$ and $\alpha = (1+r) > 1$. This appendix shows how the conditional variance term δ leads to multiple equilibria and the conditions under which the results of Section 3.4 continue to hold. To shorten the length of this appendix, I will assume that supply is given by: $s_t = A(L)\varepsilon_{1t}$, and that traders see only the private signal and the coupon stream, $\Omega_i^t = \{v_{it-j}, c_{t-j} : j > 0\}$. Adding the second component of the supply process does not change the results qualitatively.

Consider the same informational setup of as before with the signal system for agent $i < k$ given by

$$\begin{bmatrix} v_{it} \\ c_t \end{bmatrix} = \begin{bmatrix} \mathcal{J}_1(L) & 0 \\ 0 & C(L) \end{bmatrix} \begin{bmatrix} \xi_{it} \\ u_t \end{bmatrix}$$

Invoking Assumption 1 implies the derivations follow exactly as in Section 3.4, with the only difference being the addition of the conditional variance term, δ. Given that agents initially only condition on the information extracted from signals and that they are able to match the serial correlation properties of the actual price process ($p_t = D(L)\varepsilon_{1t} + G(L)u_t$), the conditional variance terms will be identical across groups in equilibrium, i.e.,

$$\begin{aligned} \delta &= \text{var}_t(p_{t+1}) + \text{var}_t(c_{t+1}) + 2\text{cov}_t(p_{t+1}, c_{t+1}) \\ &= D_0^2 \sigma_{\varepsilon_1}^2 + (G_0 + C_0)^2 \sigma_u^2. \end{aligned}$$

The corresponding market-clearing condition is given by

$$\begin{aligned} \varphi p_t &= \int_0^k \mathbb{E}_t^i(p_{t+1} + c_{t+1}) di + \int_k^1 \mathbb{E}_t^j(p_{t+1} + c_{t+1}) dj - \delta \gamma s_t \\ \varphi[D(L)\varepsilon_{1t} + G(L)u_t] &= kL^{-1}[D(L) - D_0]\frac{\varepsilon_{1t}\chi_1}{1 - \lambda_1 L} + (1-k)L^{-1}[D(L) - D_0]\frac{\varepsilon_{1t}\chi_2}{1 - \lambda_2 L} \\ &\quad + L^{-1}[G(L) - G_0]u_t + L^{-1}[C(L) - C_0]u_t - \delta\gamma A(L)\varepsilon_{1t} \end{aligned}$$

where now $\varphi = [\gamma\delta\xi + \alpha] > 1$. As before, both $D(z)$ and $G(z)$ must be analytic on the unit disk for all realizations of ε_{1t} and u_t, but the inclusion of the conditional variance has linked the two power series, which can be seen by rearranging and expanding the conditional variance term,

$$\begin{aligned} & D(z)[-\lambda_1\lambda_2\varphi z^3 + (\lambda_2 + \lambda_1)\varphi z^2 - ((1-k)\lambda_1\chi_2 + \lambda_2\chi_1 k + \varphi)z + k\chi_1 + (1-k)\chi_2] \\ &= D_0[(1-\lambda_2 z)k\chi_1 + (1-\lambda_1 z)(1-k)\chi_2] + z(1-\lambda_1 z)(1-\lambda_2 z)\gamma[D_0^2\sigma_{\varepsilon_1}^2 + (G_0 + C_0)^2\sigma_u^2]A(z) \end{aligned} \tag{31}$$

$$G(z)(1 - \varphi z) = G_0 + C_0 - C(z) \tag{32}$$

There exists a potential pole at $z = |\varphi^{-1}| < 1$ in (32) unless the free parameter G_0 is set to remove the singularity. That is, $G_0 = C(\varphi^{-1}) - C_0$ and $G^*(z) = \frac{C(\varphi^{-1}) - C(z)}{1 - \varphi z}$. The root condition for uniqueness in (31) continues to be identical to the previous section ((24)), and we know that there exists exactly one root (θ) that lies inside the unit circle. Then as before, D_0 must be set to remove the singularity at $z=\theta$. Thus we have

$$D_0[(1-\lambda_1\theta)(1-k)\chi_2+(1-\lambda_2\theta)k\chi_1]+[D_0^2\sigma_{\varepsilon_1}^2+C(\varphi^{-1})^2\sigma_u^2]\gamma A(\theta)\theta(1-\lambda_1\theta)(1-\lambda_2\theta)=0 \quad (33)$$

Given that there are two roots that satisfy (33), there are *exactly* two equilibria. Moreover, the excess 'noise' associated with the stochastic coupon stream coupled with risk-averse investors severely diminishes the probability of finding equilibria that are real valued. In order for D_0 (and the corresponding price process) to be real valued, restrictions must be placed on the parameter values. Removing uncertainty in the coupon stream ($\sigma_u^2=0$) yields two equilibria with[23]

$$D_0^*=\begin{cases}-\frac{(1-\lambda_1\theta)(1-k)\chi_2+(1-\lambda_2\theta)k\chi_1}{\sigma_{\varepsilon_1}^2\gamma A(\theta)\theta(1-\lambda_1\theta)(1-\lambda_2\theta)}\\ 0\end{cases} \quad (34)$$

The second case implies $p_t=G^*(L)u_t$ for all t. However, if $\sigma_u^2\neq 0$, then, from the quadratic formula, a necessary and sufficient condition for finding a real solution is given by the following restriction

$$\sigma_u^2\leq\frac{[(1-\lambda_1\theta)(1-k)\chi_2+(1-\lambda_2\theta)k\chi_1]^2}{4\sigma_{\varepsilon_1}^2\gamma^2A(\theta)^2\theta^2(1-\lambda_1\theta)^2(1-\lambda_2\theta)^2C(\varphi^{-1})^2}. \quad (35)$$

Assuming σ_u^2 satisfies the above restriction then the number of roots in D_0 satisfying (33) will be the number of equilibria encountered. While the potential existence of multiple equilibria is disconcerting, we are more concerned here with the revelation properties of the price process(es). The candidate equilibrium price process(es) will be given by: $p_t=D(L)^*\varepsilon_{1t}+G(L)^*u_t$, where

$$D(L)^*=\frac{D_0^*[(1-\lambda_2L)k\chi_1+(1-\lambda_1L)(1-k)\chi_2]+L(1-\lambda_1L)(1-\lambda_2L)\gamma(D_0^{*2}\sigma_{\varepsilon_1}^2+C(\varphi^{-1})^2\sigma_u^2)A(L)}{-\lambda_1\lambda_2\varphi L^3+(\lambda_2+\lambda_1)\varphi L^2-((1-k)\lambda_1\chi_2+\lambda_2\chi_1k+\varphi)L+k\chi_1+(1-k)\chi_2} \quad (36)$$

$$G(L)^*=\frac{C(\varphi)^{-1}-C(L)}{1-\varphi L}$$

The post-equilibrium observer system for trader i is then

$$\begin{bmatrix}v_{it}\\ c_t\\ p_t\end{bmatrix}=\begin{bmatrix}A(L)&0&1\\ 0&C(L)&0\\ D(L)^*&G(L)^*&0\end{bmatrix}\begin{bmatrix}\varepsilon_{1t}\\ u_t\\ \eta_{it}\end{bmatrix}$$
$$\mathbf{x}_t=\mathcal{Q}(L)\boldsymbol{\epsilon}_t \quad (37)$$

As in the previous section, this observer system will reveal the underlying shocks ε_{1t} and u_t if (37) is a fundamental Wold representation; that is, if the determinant of $\mathcal{Q}(L)$ does not contain any zeros inside the unit circle. Assuming the minimum variance condition (35) is met with equality implies that there exists exactly one root that solves (33),

$$D_0^*=-\frac{(1-\lambda_1\theta)(1-k)\chi_2+(1-\lambda_2\theta)k\chi_1}{2\sigma_{\varepsilon_1}^2\gamma A(\theta)\theta(1-\lambda_1\theta)(1-\lambda_2\theta)}.$$

[23]Singleton found these equilibria numerically.

The price process is therefore unique and given by

$$
\begin{aligned}
&D(z)^*[-\lambda_1\lambda_2\varphi z^3 + (\lambda_2+\lambda_1)\varphi z^2 - ((1-k)\lambda_1\chi_2 + \lambda_2\chi_1 k + \varphi)z + k\chi_1 + (1-k)\chi_2] \\
&= \frac{(1-\lambda_1\theta)(1-k)\chi_2 + (1-\lambda_2\theta)k\chi_1}{2\gamma\theta^2 A(\theta)(1-\lambda_1\theta)^2(1-\lambda_2\theta)^2}\Big[z(1-\lambda_1 z)(1-\lambda_2 z)A(z)[(1-\lambda_1\theta)(1-k)\chi_2 + (1-\lambda_2\theta)k\chi_1] - \\
&\qquad \theta(1-\lambda_1\theta)(1-\lambda_2\theta)A(\theta)[(1-\lambda_1 z)(1-k)\chi_2 + (1-\lambda_2 z)k\chi_1]\Big] \qquad (38)
\end{aligned}
$$

Comparing (38) with (25) it is easy to see how risk aversion enters the candidate equilibrium price process. More importantly, it is easy to see that the zeros of (25) and (38) will coincide. In other words, the zeros of the determinant of $\mathcal{Q}(L)$ are given by

$$
\begin{aligned}
\det \mathcal{Q}(L) &= -D(L)^* C(L) \\
&= -(1+\psi z)(z-\theta)[Qz^2 + Rz + S]
\end{aligned}
$$

The exact condition for price revelation encountered in Section 3.4! Appendix B shows that there are no zeros that lie inside the unit circle and therefore the price process will reveal all privately held information. The equilibrium price of this economy is determined by a symmetric information structure. Moreover as σ_u^2 falls below the minimum variance condition (35), the economy converges to the candidate equilibria with D_0^* given by (34). Using the techniques of this paper, it is easy to show that these candidate equilibria will also reveal and degenerate into symmetric information equilibria.

Appendix B: Proof of Propositions

Proof of Proposition 2

The following theorem will be used to show that two roots of (24) lie outside the unit circle with one real root inside the unit circle (see, [13]).[24]

Theorem A (Rouché)**.** *Suppose that two functions f and g are analytic inside and on a simple closed contour C. If $|f(z)| > |g(z)|$ at each point on C, then the functions $f(z)$ and $f(z)+g(z)$ have the same numbers of zeros, counting multiplicities, inside C.*

To determine the number of roots of the cubic equation

$$-\lambda_1\lambda_2\varphi z^3 + (\lambda_2+\lambda_1)\varphi z^2 - (\lambda_2\chi_1 k + (1-k)\lambda_1\chi_2 + \varphi)z + k\chi_1 + (1-k)\chi_2 \qquad (39)$$

interior to the circle $|z| = 1$, rewrite (39) as

$$-a_3 z^3 + a_2 z^2 - a_1 z + a_0$$

and define

$$f = -a_1 z + a_0, \qquad g = -a_3 z^3 + a_2 z^2$$

[24] Thanks to Ken Kasa for suggesting Rouché's Theorem in proving Proposition 3.1.

Notice that f has a (single) root inside the unit circle, since $a_1 > a_0$. To show this define $\vartheta_i = \sigma_{\varepsilon_1}/\sqrt{\sigma^2_{\varepsilon_1} + \sigma^2_{\eta_i}(1-\rho)^2}$ and $\chi_i = \vartheta_i(1-\lambda_i)$ for $i = 1, 2$, and note

$$k\chi_1(1-\lambda_2) + (1-k)\chi_2(1-\lambda_1) = (1-\lambda_1)(1-\lambda_2)[k\vartheta_1 + (1-k)\vartheta_2] < 1 < \varphi$$

Next, notice that $|f| > |g|$ on $|z| = 1$; to wit, note that[25]

$$|-\varphi(1-\lambda_1)(1-\lambda_2)| < \varphi < |(1-\lambda_1)(1-\lambda_2)[k\vartheta_1 + (1-k)\vartheta_2] - 2\varphi|$$

implies

$$|-\varphi(1-\lambda_1)(1-\lambda_2)| < |(1-\lambda_1)(1-\lambda_2)[k\vartheta_1 + (1-k)\vartheta_2] - 2\varphi|$$

which gives the result

$$|-\varphi(\lambda_1\lambda_2 - \lambda_2 - \lambda_1)| < |(1-\lambda_1)(1-\lambda_2)[k\vartheta_1 + (1-k)\vartheta_2] - \varphi|$$

Proof of Proposition 3

Following Lemma 1, in order to prove that (27) is fundamental and reveals u_t, ε_{1t} and ε_{2t}, we must show that the determinant of $\mathcal{H}(z)$ has no zeros inside the unit circle. This implies that traders equipped with VAR analysis will be able to figure out the underlying economic shocks of the economy. Recall $B(z) = 1 + \varsigma z$ where $|\varsigma| < 1$. By direct calculation, the numerator of det $\mathcal{H}(z)$ is

$$-(1+\varsigma z)(z-\theta)[Qz^2 - Rz + S]$$

where

$$\begin{aligned} Q &= \lambda_1\lambda_2[k\chi_1(1-\lambda_2\theta) + \chi_2(1-k)(1-\lambda_1\theta)](1-\theta\rho) \\ R &= (1-\lambda_1\theta)(1-\lambda_2\theta)[k\chi_1\lambda_2 + (1-k)\chi_2\lambda_1] + (1-\theta\rho)[k\chi_1\lambda_1(1-\lambda_2\theta) + (1-k)\chi_2\lambda_2(1-\lambda_1\theta)] \\ S &= (1-\lambda_1\theta)(1-\lambda_2\theta)[k\chi_1 + (1-k)\chi_2]. \end{aligned}$$

The determinant of $\mathcal{H}(z)$ has four zeros. The root $z = \theta$ is by construction (recall D_0^* was set to ensure such a numerator zero would cancel the like term in the denominator of $D(z)$) and the root $z = -\psi^{-1}$ lies outside the unit circle. Therefore, the roots of

$$Qz^2 + Rz + S \tag{40}$$

determine whether or not representation (27) is a fundamental (Wold) moving average. We must now show that for $\varphi > 1$ there exists no roots that lie inside the unit circle. In other words, the quadratic given by (40) has both roots outside of the unit circle. This implies that the representation (27) is fundamental, hence eliminating the need to forecast the forecasts of others. Recall the

[25] It is interesting to note in passing that given $\lambda_1 = \lambda_2$ the above inequalities can be established via the unique properties associated with the golden ratio (i.e., the positive number that solves the equation $\phi + \phi^{-1} = 1$).

quadratic is given by

$$Y(z) = Qz^2 - Rz + S$$
$$Q = \lambda_1\lambda_2[k\chi_1(1-\lambda_2\theta) + (1-k)\chi_2(1-\lambda_1\theta)](1-\theta\rho)$$
$$R = (1-\lambda_1\theta)(1-\lambda_2\theta)[k\chi_1\lambda_2 + (1-k)\chi_2\lambda_1] + (1-\theta\rho)[k\chi_1\lambda_1(1-\lambda_2\theta) + (1-k)\chi_2\lambda_2(1-\lambda_1\theta)]$$
$$S = (1-\lambda_1\theta)(1-\lambda_2\theta)[k\chi_1 + (1-k)\chi_2]$$

First notice that Q, R and S are all strictly positive, implying that $Y(0) > 0$ and all the corresponding roots must be strictly positive. Moreover $Y(1) > 0$; ignoring $(1-k)\chi_2$ terms (due to symmetry) yields

$$\begin{aligned} Y(1) &= k\chi_1[\lambda_1\lambda_2(1-\lambda_2\theta)(1-\theta\rho) - (1-\lambda_1\theta)(1-\lambda_2\theta)\lambda_2 - (1-\theta\rho)\lambda_1(1-\lambda_2\theta) + (1-\lambda_1\theta)(1-\lambda_2\theta)] \\ &= k\chi_1(1-\lambda_2\theta)(1-\lambda_2)[(1-\lambda_1\theta) - \lambda_1(1-\theta\rho)] \end{aligned}$$

which is clearly positive because $(1-\lambda_1\theta) > \lambda_1(1-\theta\rho)$.

We now need to rule out the case that both roots lie inside the unit circle. We do this by showing that the minimum lies outside the unit circle.

$$z^* = \text{argmin } Y(z) = \frac{R}{2Q} > 1$$

Again focusing on $k\chi_1$, we need to show that

$$\begin{aligned} R - 2Q &= k\chi_1\lambda_2(1-\lambda_1\theta)(1-\lambda_2\theta) + (1-\theta\rho)k\chi_1\lambda_1(1-\lambda_2\theta) - 2[\lambda_1\lambda_2 k\chi_1(1-\lambda_2\theta)(1-\theta\rho)] > 0 \\ &= k\chi_1(1-\lambda_2\theta)\{\lambda_2[(1-\lambda_1\theta) - \lambda_1(1-\rho\theta)] + (1-\rho\theta)\lambda_1(1-\lambda_2)\} \end{aligned}$$

which is clearly positive.

References

[1] K. Adam, Optimal monetary policy with imperfect common knowledge, J. Monet. Econ., forthcoming.

[2] A.R. Admati, A noisy rational expectations equilibrium for multi-asset securities markets, Econometrica 53 (3) (1985) 629–658.

[3] F. Allen, S. Morris, H.S. Shin, Beauty contests and iterated expectations in asset markets, Rev. Finan. Stud. (19) 3 (2006) 719–752.

[4] G.M. Angeletos, A. Pavan, Efficient use of information and welfare analysis in economies with complementarities and asymmetric information, manuscript, MIT, 2005.

[5] P. Bacchetta, E. van Wincoop, Can information heterogeneity explain the exchange rate puzzle?, NBER working paper: No. 9498, 2003.

[6] P. Bacchetta, E. van Wincoop, Can information heterogeneity explain the exchange rate puzzle?, Amer. Econ. Rev. 3 (96) (2006) 552–576.

[7] P. Bacchetta, E. van Wincoop, Higher order expectations in asset pricing, manuscript, Studienzentrum Gerzensee, 2006.

[8] D. Bernhardt, P. Seiler, B. Taub, Speculative Dynamics, manuscript, University of Illinois, 2006.

[9] O.J. Blanchard, C.M. Kahn, The solution of linear difference models under rational expectations, Econometrica (48) 5 (1980) 1305–1312.

[10] W.A. Brock, C.H. Hommes, A rational route to randomness, Econometrica (65) 5 (1997) 1059–1095.

[11] W.A. Brock, C.H. Hommes, Heterogeneous beliefs and routes to chaos in a simple asset pricing model, J. Econ. Dynam. Control 22 (1998) 1235–1274.

[12] P.J. Brockwell and R.A. Davis, Time Series: Theory and Methods, Springer Verlag, New York, NY, 1991.

[13] R.V. Churchill, J.W. Brown, Complex Variables and Applications, McGraw-Hill, New York, NY, 1990.

[14] F.D. Foster, S. Viswanathan, Strategic trading when agents forecast the forecasts of others, J. Finance 51 (1996) 1437-1478.

[15] C.A. Futia, Rational expectations in stationary linear models, Econometrica 49 (1) (1981) 171-192.

[16] L.P. Hansen, T.J. Sargent, Formulating and estimating dynamic linear rational expectations models, J. Econ. Dynam. Control 2 (1980) 7-46.

[17] H. He, J. Wang, Differential information and dynamic behavior of stock trading volume, Rev. Finan. Stud. 8 (1995) 919-972.

[18] C. Hellwig, L. Veldkamp, Knowing what others know: coordination motives in information acquisition, manuscript, UCLA, 2005.

[19] K. Kasa, Forecasting the forecasts of others in the frequency domain, Rev. Econ. Dynam. 3 (2000) 726-756.

[20] K. Kasa, T.B. Walker, C.H. Whiteman, Asset prices in a time series model with perpetually disparately informed, competitive traders, manuscript, University of Iowa, 2006.

[21] J.M. Keynes, The General Theory of Employment, Interest and Money, Macmillan Press, London, 1936.

[22] A. Kolmogorov, Stationary sequences in Hilbert space, Bull. Math. Univ. Moscow 2 (6) (1941) 1–40.

[23] R.E. Lucas, An equilibrium model of the business cycle, J. Polit. Economy 83 (1975) 1113-1144.

[24] S. McCafferty, R. Driskill, Problems of existence and uniqueness in nonlinear rational expectations models, Econometrica 48 (1980) 1313-1317.

[25] S. Morris, H.S. Shin, Global games: theory and applications, in: M. Dewatripont, L. Hansen, S. Turnovsky (Eds.), Advances in Economics and Econometrics: Theory and Applications, Cambridge University Press, Cambridge, 2003, pp. 56–115.

[26] J.F. Muth, Optimal properties of exponentially weighted forecasts, J. Amer. Statistical Assoc. 55 (1960) 299-306.

[27] K.P. Nimark, Dynamic pricing, imperfect common knowledge and inflation inertia, ECB Working Paper No. 474, 2005.

[28] J.G. Pearlman, D. Currie, P. Levine, Rational expectations models with private information, Econ. Modelling 3 (1986) 90-105.

[29] J.G. Pearlman, T.J. Sargent, Knowing the forecasts of others, Rev. Econ. Dynam. 8 (2005) 480-497.

[30] E.S. Phelps, The trouble with rational expectations and the problem of inflation stabilization, in: E.S. Phelps, R. Frydman (Eds.), Individual Forecasting and Aggregate Outcomes, Cambridge University Press, Cambridge, 1983, pp. 31-45.

[31] M. Pourahmadi, Foundations of Time Series Analysis and Prediction Theory, Wiley-Interscience, New York, NY, 2001.

[32] T.J. Sargent, Interpreting economic time series, J. Polit. Economy, 89, (1981) 213-218.

[33] T.J. Sargent, Macroeconomic Theory, Second Edition, Academic Press, London, 1987.

[34] T.J. Sargent, Equilibrium with signal extraction from endogenous variables, J. Econ. Dynam. Control 15 (1991) 245-273.

[35] K.J. Singleton, Asset prices in a time series model with disparately informed, competitive traders, in: W. Barnett, K. Singleton (Eds.), New Approaches to Monetary Economics, Cambridge University Press, Cambridge, 1987, pp. 249–272.

[36] R.M. Townsend, Forecasting the forecasts of others, J. Polit. Economy 91 (1983) 546-588.

[37] T. Ui, A note on the lucas model: iterated expectations and the neutrality of money, manuscript, Yokohama University, 2003.

[38] T.B. Walker, Multiple equilibria in a simple asset pricing model, manuscript, Indiana University, 2006.

[39] J. Wang, A model of asset prices under asymmetric information, Rev. Econ. Stud. 60 (1993) 249-282.

[40] J. Wang, A model of competitive stock trading volume, J. Polit. Economy 102 (1994) 127-168.

[41] C.H. Whiteman, Linear Rational Expectations Models: A User's Guide, University of Minnesota Press, Minneapolis, 1983.

[42] P. Whittle, Prediction and Regulation by Linear Least-Square Methods, University of Minnesota Press, Minneapolis, 1983.

[43] M. Woodford, Imperfect common knowledge and the effects of monetary policy, in: P. Aghion, R. Frydman, J. Stiglitz, M. Woodford (Eds.), Knowledge, Information, and Expectations in Modern Macroeconomics, Princeton University Press, Princeton, N.J., 2003, pp. 25–59.