A Note on Futia (1981)'s Non-Existence Pathology of Rational Expectations Equilibria*

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Abstract

We resolve the non-existence pathologies of dynamic rational expectations equilibria attributed to signal extraction from endogenous variables first discovered by Futia (1981). Non-existence is overturned once it is recognized that rational agents take into account the structure of the model when generating equilibrium outcomes. We show that where Futia (1981) thought an equilibrium did not exist, a Rational Expectations equilibrium identical to the Full Communication equilibrium does exist, thereby resolving the long-standing non-existence pathology.

Keywords: Rational Expectations, Incomplete Information, Information Equilibrium

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1 Introduction

In a seminal paper in the January 1981 issue of *Econometrica*, Carl Futia characterized rational expectations equilibria in the presence of incomplete information and where equilibrium variables provided endogenous information to economic agents. The problem is non-trivial since the rational expectations equilibrium requires finding a fixed point in both optimal strategies and information sets. In standard representative agent economies these fixed point conditions are well understood and easy to compute. When agents have incomplete information and extract information from endogenous variables, the fixed point conditions are more challenging to construct.[1]

The final message of Futia (1981) was mixed. On the one hand, Futia showed in the context of a simple model that a rational expectations equilibrium where all the agents are symmetrically informed is always identical to the unique Full Communication (FC) equilibrium, (i.e. the equilibrium that would emerge if all the available information in the economy was pooled and given exogenously to the agents). On the other hand, he also showed that this is true only if the stochastic process characterizing the FC equilibrium obeys a certain invertibility property. When the invertibility property is not satisfied, he argued that there cannot be a rational expectations equilibrium and hence the non-existence pathology. The non-existence pathology undoubtedly casts a feeling of uneasiness in dealing with dynamic rational expectations models with incomplete information. Arguably, the response of the subsequent literature to the troubling message of Futia (1981) has been to structure the information set with as much exogenous information as necessary to side-step the existence issue [Atkeson (2000)]. In so doing, information is taken out of the hands of the equilibrium interactions and put into the hands of mechanical signal extraction algorithms. This modeling choice, while ensuring existence, has the unintended consequence of mooting potentially interesting insights stemming from the subtle ways in which information is shaped by equilibrium forces.[2]

In this paper we show that the non-existence pathology in Futia (1981) is a consequence of an incon-

[1] Townsend (1983) and Sargent (1991) contain informative discussions of the issues.

[2] See Rondina and Walker (2012a) for a systematic analysis of how the informational feedback in equilibrium can generate a new class of rational expectations equilibria.

sistency between the definition of rational expectations equilibria employed in Futia's analysis and the argument he used in proving that a rational expectations equilibrium must always be identical to the FC equilibrium. We show that the definition of a rational expectations equilibrium stated by Futia prevents rational agents from making use of the knowledge that the variables that they observe are the outcome of an equilibrium process. We argue that this is inconsistent with the standard definition of rational expectations. More precisely, in the "only if" direction of Futia's main existence argument, the agents act according to a *reduced-form* interpretation of the observed equilibrium variables; while in the "if" direction an argument is used that relies on a *structural* interpretation of the observed equilibrium variables on the part of the agents. The non-existence pathology originates from the "only if" part of the proof. Once the definition of the rational expectations equilibrium is modified to address such inconsistency, the non-existence pathology disappears. Where Futia (1981) thought an equilibrium did not exist, a Rational Expectations equilibrium identical to the Full Communication equilibrium does exist.

The broader message of this note is that in the presence of signal extraction from equilibrium variables, the fixed point of information that is part of any rational expectations characterization poses subtle challenges to the modeler, both at the definition and at the solution stages. However by resolving the existence pathologies of Futia, this paper pushes the literature forward by arguing that more information can be safely entrusted to equilibrium interactions, with the potential of gaining useful insights.

2 Futia (1981): Framework and Main Results

Section 1 of Futia (1981) describes the model and is titled, "Land Speculation in Hilbert Space." He assumed that there is a fixed quantity of land with speculative and non-speculative traders. The non-speculative demand for land at each date t is a random variable (i.e., noise traders) that never exceeds the total supply. The difference between total supply and and non-speculative demand is denoted s_t. The speculative demand of trader i for land arises from demand function $q_t^i = \mathbb{E}_t^i p_{t+1} - \alpha p_t$, where

$\alpha > 1$ is the opportunity cost of funds.[3]

Setting supply equal to demand delivers the forward looking linear expectational difference equation[4]

$$p_t = \beta \mathbb{E}\big[p_{t+1}|\Omega_t\big] + s_t \tag{2.1}$$

where $\alpha^{-1} = \beta < 1$, s_t is a covariance stationary process of the form $s_t = A(L)\varepsilon_t$ with $A(L)$ a square summable lag polynomial in non-negative powers of L and $\varepsilon_t \sim N(0, \sigma_\varepsilon)$ idiosyncratically distributed across time t. Expectations are assumed to be rational, which results in the expectation operator corresponding to linear projection upon the space defined by the information set; Ω_t is the information set available at time t of the representative agent. In this note, following Futia (1981) we restrict our attention to the case of symmetric information sets, i.e. a situation where all agents have access to the same, possibly incomplete, information. In terms of notation, we denote $\mathbb{V}_t(x)$ as the smallest closed linear subspace spanned by the infinite history of the random variable x_t up to time t, namely $x^t \equiv \{x_t, x_{t-1}, x_{t-2}, ...\}$. In what follows we will also assume that the polynomial $A(L)$ is invertible, so that observing the history of s_t is equivalent (in mean square norm) to observing the history of ε_t, formally $\mathbb{V}_t(s) = \mathbb{V}_t(\varepsilon)$.[5]

We next present the key steps in Futia (1981)'s existence results.

Full Communication Equilibrium

The first step consists of defining and solving for a Full Communication equilibrium (FC) as introduced by Radner (1979). It corresponds to solving the expectational equation under the assumption that the agents observe the history of innovations ε_t up to time t, i.e. $\Omega_t = \mathbb{V}_t(\varepsilon)$. The formal definition follows.

Definition FC. *A full communication equilibrium is a stationary process for p_t such that for $t \in \mathbb{Z}$*

[3] Rondina and Walker (2012a) derive the microfoundations that motivate the model of Futia (1981).

[4] Without loss of generality, we have suppressed the constant β multiplying s_t.

[5] Relaxing this assumption opens up a set of interesting implications for the dynamics of a rational expectations equilibrium (see Rondina and Walker (2012a)); for the purpose of this paper it is however not essential and so we abstract from it.

1. $\mathbb{V}_t(p) \subseteq \mathbb{V}_t(\varepsilon)$

2. $p_t = \beta\mathbb{E}\big[p_{t+1}|\mathbb{V}_t(\varepsilon)\big] + s_t$

The following proposition states that in the present setting there always exists a unique FC equilibrium and provides a closed form solution.

Proposition 1. *A FC equilibrium exists, is unique, and is given by*

$$p_t = \left(\frac{LA(L) - \beta A(\beta)}{L - \beta}\right)\varepsilon_t. \tag{2.2}$$

Proof. See Appendix. □

Restating Futia (1981)

One of the main contributions of Futia (1981) is the arguement that a relevant notion of a rational expectations equilibrium should take into account that agents observe variables that are themselves equilibrium outcomes, or, put differently, that the information set of agents arises endogenously. In the univariate context of the model (2.1) this corresponds to the assumption that the representative agent obtains information only from current and past equilibrium prices. Formally, the information set in (2.1) is specified as $\Omega_t = \mathbb{I}_t(p)$, to denote that the only source of information is transmitted by prices. Futia's definition of an equilibrium specifies a particular mathematical structure for $\mathbb{I}_t(p)$.

Definition FRE. *A Futia rational expectations (FRE) equilibrium is a stationary process p_t such that for $t \in \mathbb{Z}$*

1. $\mathbb{V}_t(p) \subseteq \mathbb{V}_t(\varepsilon)$

2. $p_t = \beta\mathbb{E}\big[p_{t+1}|\mathbb{I}_t(p_t)\big] + s_t \quad$ *where* $\quad \mathbb{I}_t(p) = \mathbb{V}_t(p)$

The endogeneity of the information set makes the definition more difficult to operationalize, in the sense that it is not immediately clear how to translate it into a closed form solution for p_t. One way to proceed is to study conditions under which an equilibrium with endogenous information is equivalent to an equilibrium with exogenous information, the latter being easier to characterize in closed form. To this end, Futia presents a result that ensures that any FRE equilibrium, when it exists, takes the form of a FC equilibrium.

Proposition 2. *A FRE equilibrium for* (2.1) *when it exists is identical to the FC equilibrium.*

The proof of the proposition is as follows. When a FRE exists, *along* the equilibrium path the representative agent must be able to compute

$$s_t = p_t - \beta\mathbb{E}\big[p_{t+1}|\mathbb{V}_t(p)\big] \tag{2.3}$$

which means that in any FRE equilibrium the price reveals the history of s_t, or, equivalently, of ε_t. Therefore, in any FRE equilibrium one must have

$$\mathbb{E}\big[p_{t+1}|\mathbb{V}_t(p)\big] = \mathbb{E}\big[p_{t+1}|\mathbb{V}_t(\varepsilon)\big] \tag{2.4}$$

where the left hand side is the expectational operator of a FC equilibrium. Hence, if a FRE equilibrium exists it must be identical to a FC equilibrium.

3 Non-Existence Pathology

The above proposition suggests an easy way to solve for a FRE equilibrium, provided one exists: take the price process of the FC equilibrium and check that (2.4) holds, or, equivalently, that $\mathbb{V}_t(p) = \mathbb{V}_t(\varepsilon)$. Futia, however, noticed that there are instances in which the FC equilibrium price does not satisfy (2.4). Because any FRE equilibrium must be a FC equilibrium, the logical conclusion is that in those instances a FRE does not exist. Following Futia, let the process s_t take a simple $MA(1)$ form

$$s_t = \varepsilon_t + \theta\varepsilon_{t-1} \tag{3.1}$$

with $|\theta| < 1$ so that $\mathbb{V}_t(s) = \mathbb{V}_t(\varepsilon)$. Using proposition 1 the FC equilibrium for this specification of $A(L)$ is given by

$$p_t = (1 + \beta\theta)\left(\varepsilon_t + \frac{\theta}{1 + \beta\theta}\varepsilon_{t-1}\right). \tag{3.2}$$

Futia remarked that for values of β and θ such that $|\frac{\theta}{1+\beta\theta}| > 1$ there does not exist a square-summable linear combination of current and past prices that reveals ε_t, which means that $\mathbb{V}_t(p) \subset \mathbb{V}_t(\varepsilon)$. From proposition 2 the price p_t cannot possibly be a FRE. Hence, a FRE does not exist. This would be the case, for example, for values of β close to 1 and for $\theta = -5/8$, which are the values considered by Futia. The following proposition states the non-existence result for a general $A(L)$.

Proposition 3. *Let $\tilde{P}(L)/\tilde{Q}(L)$ be the ratio of lag polynomials obtained by canceling the factor $L - \beta$ in the FC equilibrium mapping $\frac{LA(L)-\beta A(\beta)}{L-\beta}$. If $\tilde{P}(L)$ contains a zero inside the unit circle a FRE equilibrium for (2.1) does not exist.*

The non-existence result is clearly troubling as it happens for values in the parameter space that do not carry a significance that can be easily recognize. For example, for the $MA(1)$ example above, if one were to set $\theta = -3/8$ existence of a FRE equilibrium would be reinstanted. However, the process for s_t under the two alternative values for θ does not present any significant differences.

4 Information Equilibrium

In this section we show that a closer look at Futia's analysis reveals that the definition of a Futia Rational Expectations equilibrium and a critical step in the proof of proposition 2, i.e. that a FRE equilibrium must always be equal to a FC equilibrium, display a subtle but crucial inconsistency. The proof of Proposition 2 states that under any FRE the representative agent must be able to recover s_t by performing the computation

$$s_t = p_t - \beta\mathbb{E}(p_{t+1}|\mathbb{I}_t(p)). \tag{4.1}$$

In order to derive s_t from the history of prices p^t the representative agent is assumed to be using her knowledge that the equilibrium equation takes the form specified in point 2 of the Definition FRE. In other words, the rational agent combines the observed history of prices with her knowledge of the model that generates such history in equilibrium. Taking into account this knowledge suggests that among all the possible square-summable linear combinations of the elements in p^t there is a particular one that the agent can interpret only by considering the structural equation that generated it. Such structural interpretation results into recognizing that s_t is given by (4.1). The definition of a Futia Rational Expectations equilibrium is, however, inconsistent with the structural interpretation used in proving proposition 2. The FRE definition imposes $\mathbb{I}_t(p_t) = \mathbb{V}_t(p_t)$, and in so doing forces the agents to ignore (4.1). For the specific example described above, this implies that the FRE is a limited-information rational expectations equilibrium. The reason is that in taking the linear projection of p_{t+1} onto $\mathbb{V}_t(p_t)$ the information is organized according to the Wold representation of p_t, which, by definition, is invariant to the structural model as long as the autocovariance generating function of p_t remains the same. However, the structural model imposes restrictions in the relationship between the autocovariance of p_t and the structural innovations ε_t that are being ignored in the FRE. Such restrictions are sometimes labeled "cross-equation restrictions" and are what Hansen and Sargent (1980) call the "hallmark" of rational expectations.

The information set $\mathbb{V}_t(p)$ summarizes the reduced-form information contained in the history of p_t, but it ignores that rational agents know the structural model and can use such information. To capture the use of the structural knowledge in extracting information from the equilibrium p_t, we define the information flow from the model with the notation $\mathbb{M}_t(p)$. $\mathbb{M}_t(p)$ formalizes the idea that the history of p_t combined with the knowledge of the model represents an independent source of information in models of rational expectations with incomplete information.

Futia's proof of proposition 2 is reconciled with the definition of a rational expectations equilibrium once the information set $\mathbb{M}_t(p)$ is made part of $\mathbb{I}_t(p)$. We call the modified definition a rational expectations *Information Equilibrium.*

Definition IE. *A rational expectations Information Equilibrium (IE) is a stationary process p_t such* *that for* $t \in \mathbb{Z}$

1. $\mathbb{V}_t(p) \subseteq \mathbb{V}_t(\varepsilon)$

2. $p_t = \beta\mathbb{E}\big[p_{t+1}|\mathbb{I}_t(p_t)\big] + s_t$ *where* $\mathbb{I}_t(p_t) = \mathbb{V}_t(p) \vee \mathbb{M}_t(p)$

The notation $\vee$ stands for the smallest linear subspace containing all elements of the two subspaces. We are now ready to state the main result.

Proposition 4. *A rational expectations Information Equilibrium for* (2.1) *always exists and is identical to the FC equilibrium.*

The problem of non-existence for the FRE equilibrium was related to the possibility that the FC equilibrium price, once used as the candidate equilibrium price, would contain strictly less information than $\mathbb{V}_t(\varepsilon)$. The Information Equilibrium is not subject to this failure because any candidate equilibrium price *always* reveals the history of s_t, and thus ε_t, to the rational agent through the information $\mathbb{M}_t(p)$. Formally, in any candidate equilibrium $\mathbb{I}_t(p) = \mathbb{V}_t(s)$, which is then always true when p_t is specified as the FC equilibrium. Hence, once the definition of a rational expectations equilibrium is adjusted to be consistent with the proof of proposition 2, the non-existence pathology of Futia (1981) disappears: a rational expectations equilibrium always exists and it is equal to the Full Communication equilibrium.

5 Conclusion

In this note we have shown that the non-existence pathology of rational expectations equilibria with incomplete information remarked by Futia (1981) is the consequence of an inconsistency between Futia's definition of a rational expectations equilibrium and the key argument used to prove that a rational expectations equilibrium is always equal to a Full Communication equilibrium. Once the definition of a rational expectations equilibrium is modified to eliminate the inconsistency, Futia (1981)'s non-existence pathology disappears. We conclude by pointing out that the resolution of the pathology carries a broader message to researchers working with rational expectations models under incomplete information and dispersed information. Recognizing that the model itself is a source of information once combined with the equilibrium outcome can have very important consequences for the nature of the resulting equilibrium. We address this point more systematically in Rondina and Walker (2012a) and Rondina and Walker (2012b).

6 Appendix

Proof of Proposition 1 In a FC equilibrium $p_t = P(L)\varepsilon_t$, where $P(L)$ is a square summable polynomial in non-negative powers of L, the one-period ahead expectations can be written as

$$\mathbb{E}\big[p_{t+1}|\mathbb{V}_t(\varepsilon)\big] = \big[P(L) - P(0)\big]L^{-1}\varepsilon_t. \tag{6.1}$$

Substituting this expression into the equilibrium equation and using the Reisz-Fischer Theorem provides a functional equation for $P(z)$ in z-transform according to

$$zP(z) = \beta\big[P(z) - P(0)\big] + zA(z). \tag{6.2}$$

The choice of $P(0)$ is done so to ensure that the solution polynomial has no unstable autoregressive root at $z = \beta$, which corresponds to setting $P(0) = A(\beta)$ [see Whiteman (1983)]. The result of the proposition then immediately follows. An alternative way to prove the proposition is to recognize that the equilibrium equation can be written as the discounted sum of expectations about current and future s_t's and then apply the Hansen-Sargent formula to characterized the summation, as derived in Hansen and Sargent (1980).

References

Atkeson, A. (2000): "Comment on 'Rethinking Multiple Equilbria in Macroeconomic Modeling'," in *NBER Macroeconomics Annual 2000*, ed. by B. S. Bernanke, and K. Rogoff. MIT Press.

Futia, C. A. (1981): "Rational Expectations in Stationary Linear Models," *Econometrica*, 49(1), 171–192.

Hansen, L. P., and T. J. Sargent (1980): "Formulating and Estimating Dynamic Linear Rational Expectations Models," *Journal of Economic Dynamics and Control*, 2, 7–46.

Radner, R. (1979): "Rational Expectations Equilibrium: Generic Existence and the Information Revealed by Prices," *Econometrica*, 47(3), 655–678.

Rondina, G., and T. B. Walker (2012a): "Information Equilibria in Dynamic Economies with Dispersed Information," Working Paper.

——— (2012b): "Informational Fragility of Dynamic Rational Expectations Equilibria," Working Paper.

Sargent, T. J. (1991): "Equilibrium with Signal Extraction from Endogenous Variables," *Journal of Economic Dynamics and Control*, 15, 245–273.

Townsend, R. M. (1983): "Forecasting the Forecasts of Others," *Journal of Political Economy*, 91, 546–588.

Whiteman, C. (1983): *Linear Rational Expectations Models: A User's Guide*. University of Minnesota Press, Minneapolis.